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The discrete Painlevé I equations: transcendental integrability and asymptotic solutions

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Abstract

The integrability of the discrete Painlevé I equation (dP-I) is reviewed and its integrability studied. We establish the existence of a conserved quantity which is algebraic in the case of the autonomous dP-I equation and it is argued that the non-autonomous dP-I map has a non-algebraic invariant. Our analysis leads to, among other results the construction of asymptotic solutions with interesting structures.

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1. Introduction

Discrete nonlinear mappings have gained renewed interest in recent years, first as discretized versions of their continuous counterparts [5, 7, 22], and second as by-products of infinite discrete symmetries in two-dimensional statistical mechanics models [5,22]. This equation has appeared in two-dimensional quantum gravity [10], in the theory of orthogonal polynomials [24], as well as in the problem of counting graphs drawn on a Riemann surface [4]. They may also be considered as dynamical systems evolving in discrete time. But as discrete nonlinear equations, the question of their integrability is as central as in continuous nonlinear equations.

In this work we address ourselves to one particular class of these nonlinear mappings: the class of discrete Painlevé-1 mapping described by the dP-I equation. Of course it is known that several discretizations of the Painlevé-1 equation exist [7]. Here we shall consider the following canonical form of the dP-I equation:

$$x_{n+1} + x_n + x_{n-1} = b + \frac{c_n}{x_n} \tag{1.1}$$

which has appeared in many research topics. This is the simplest form of all the discrete Painlevé equations. It has been treated recently by many authors since it is known to be integrable [10]: it has a Lax pair and is solvable by the isomonodromic method. However, no explicit form of the solution exists and the isomonodromic method yields only an asymptotic form.

The problem of integrability was tackled about a decade ago by Grammaticos and co-workers [11–13]. To decide whether or not a discrete nonlinear equation is integrable they proposed the criterion of *singularity confinement* as an extension of the *Painlevé test*, used in continuous nonlinear equations. This test means that if the discrete nonlinear equation is integrable, iteration of the related mappings will eventually become free of singularities. However, shortly after, it was discovered by Hietarinta and Viallet that some mappings do pass the singularity confinement test but still exhibit chaotic behaviour. These authors have extended their analysis to complex projective space [9, 14, 15], and come to the conclusion that one should rather pay attention to the growth of the complexity of the iterated maps, which is described by the 'degree' of the factorized part of iterated maps.

So they are led to define the concept of *algebraic entropy* of a birational map [3]. Such a map is certainly not integrable if its algebraic entropy is not zero. But one can only conjecture that a map with zero algebraic entropy can be integrable. Hietarinta and Viallet working in a complex projective representation of discrete nonlinear maps have found strengthening arguments in favour of the algebraic entropy criterion. But recently Ohta *et al* [20] have shown that for the dP-I map the two tests are really equivalent. For the sake of completeness let us mention that Ablowitz *et al* [1] have attempted to generalize the Painlevé test for nonlinear difference equations (as opposed to discrete nonlinear equations).

Amid this intense research activity on the integrability of the dP-I equation, little attention has been devoted to knowledge of the form of the solution except in [19].

The aim of this paper is to look at the integrability of the dP-I mapping under an alternative point of view concentrated on the existence of an invariant manifold which turns out to be, as we shall see non-algebraic. A by-product of this investigation is the explicit form of an asymptotic solution. The paper is organized as follows. In section 2, we review the concept of algebraic entropy introduced by Hietarinta and Viallet and discuss some of its consequences; we also give some examples of computation of algebraic entropy. Section 3 is devoted to the structure of the autonomous dP-I mapping. This case is exactly integrable in terms of elliptic functions and has an algebraic invariant. We give the explicit parametrization in terms of Weierstrass elliptic functions and determine the periodic points of this invariant.

In section 4, we study the non-autonomous dP-I mapping as multiplicatively perturbed autonomous dP-I mapping, using the method of singularity confinement to calculate its algebraic entropy. We construct perturbatively an invariant formal series which converges locally and we show that it is not a sufficient condition which goes over a global one.

Section 5 treats the non-autonomous dP-I mapping directly. We analyse the difficulties of determining its invariant and give arguments as to why this invariant should be non-algebraic. This section is essential for the understanding of the integrability of the non-autonomous dP-I equation. What characterizes, in fact, the integrability is the conservation of areas upon going to infinity. We will show that this example is at the borderline of applicability of the KAM theorem, but may serve to extend the hypothesis of the KAM theorem to birational transformations in two variables, which upon regularization conserve areas. Finally, section 6 gives the explicit construction of the asymptotic solution, i.e. at large discrete time; this solution displays some characteristic features which may be attributed to integrability. We compare its properties with those obtained by others, in particular by Joshi [18, 19].

2. Algebraic entropy

2.1. Some useful topics of algebraic geometry

In this section, we recall some useful definitions and give some usual mathematical notation in algebraic geometry [23]. We will focus on the main notions necessary for the theoretical development of singularity confinement. Set $\mathbb{K}$ for any of the number fields (R) or (C). We call $\mathbb{K}$-affine space of dimension n, denoted by $\mathbb{K}^n$ the set of points

$$\mathbb{K}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{K}\}. \tag{2.1}$$

In this context, the relevant sets in algebraic geometry are defined as the zeros of polynomials. In fact, if we consider $\mathcal{I}$ as an ideal of the polynomial ring $\mathbb{K}[X_1, \ldots, X_n]$, the algebraic set generated by the ideal $\mathcal{I}$ is

$$V(\mathcal{I}) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{K}^n \mid P(x_1, \ldots, x_n) = 0, P \in \mathcal{I}\}. \tag{2.2}$$

As closed sets, they allow one to define a topology called the *topology of Zariski*. We give the verification of the axioms and some characteristics of this topology in appendix A.1. Among these sets, some play a particular role: the *irreducible* sets. A set Y of a topological space X is called irreducible if $Y = U \cap V$, or U and V are closed sets, then $X = U$ or $Y = V$. To this geometric notion one can associate an algebraic equivalent notion: the quotient ring $\Gamma(V] = \mathbb{K}[X_1, \ldots, X_n]/\mathcal{I}$ is integral[3].

Another class of spaces play a dominant role in algebraic geometry: these are the *projective spaces* $P^n\mathbb{K}$. Locally isomorphic to affine spaces, and completed at infinity by a set isomorphic to $P^{n-1}\mathbb{K}$, they are defined as the quotient space $\mathbb{K}^n \setminus \mathcal{R}$, where $\mathcal{R}$ denotes the following equivalence relation: $(x_0, \ldots, x_n)\mathcal{R}(y_0, \ldots, y_n)$ if there exists $\lambda \neq 0$ such that $\forall i = 0, \ldots, n$, $x_i = \lambda y_i$. An element of this equivalence class will be denoted by: $\bar{x} = (x_0 : \ldots : x_n)$. Zaraski topology is obtained by passage to the quotient. Thus algebraic projective sets are generated by zeros of homogeneous polynomials of $\mathbb{K}[X_0, \ldots, X_n]$. Now consider the open set U_i complementary to the hyperplane at infinity $x_i \neq 0$ and the mapping

$$\phi_i \begin{cases} U_i \subset \mathbb{PK}^n \longrightarrow \mathbb{K}^n \\ [x_1, \ldots, x_i, \ldots, x_n] \longmapsto [x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i]. \end{cases} \tag{2.3}$$

This is really an isomorphism since the inverse mapping is given by

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \longmapsto (x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n). \tag{2.4}$$

We refer the reader to appendix A.2 for morphisms between algebraic varieties (irreducible sets). Here we are interested in the main morphism of this paper: the *blow-up* morphism. Consider the product of projective spaces $P^n\mathbb{K} \times P^{n-1}\mathbb{K}$ and the closed subspace $\prod$ defined by the system of equations $x_i y_j = x_j y_i$, $\forall i = 1, \ldots, n$. Let $\xi = (1 : 0 : \ldots : 0)$ and call the first component projection σ. Thus for any Q such that $Q = (x_0 : \ldots : x_n) \neq \xi$ in $P^n\mathbb{K}$, there exists a unique forerunner in $\prod$ which is $\sigma^{-1}(x_0 : \ldots : x_n) = (x_0 : \ldots : x_n; x_1 : \ldots : x_n)$. But if $Q = \xi$ the pre-image of this point is the product $\xi \times P^{n-1}\mathbb{K}$. Consequently, σ^{-1} is a blow-up centred at ξ and is a regular isomorphism of $P^n\mathbb{K} \setminus \xi \longrightarrow \prod \setminus \xi \times P^{n-1}\mathbb{K}$. Using local variables, one can define this blow-up mapping anywhere else.

Let us examine what happens in the neighbourhood of ξ. Consider the straight line L of $P^n\mathbb{K}$ going through ξ and given by the equations $x_j = \alpha_j x_i$, $\forall j \neq i$. We see that, with $\alpha_i = 1$ at the i^{th} place, σ^{-1} $((x_0 : \ldots : x_n) \in L) = (x_0 : \ldots : x_n; \alpha_1 : \ldots : 1 : \ldots : \alpha_n)$, σ^{-1}

[3] A ring A is integral if $\forall (a, b) \in A^2$, $ab = 0$ then $a = 0$ or $b = 0$.

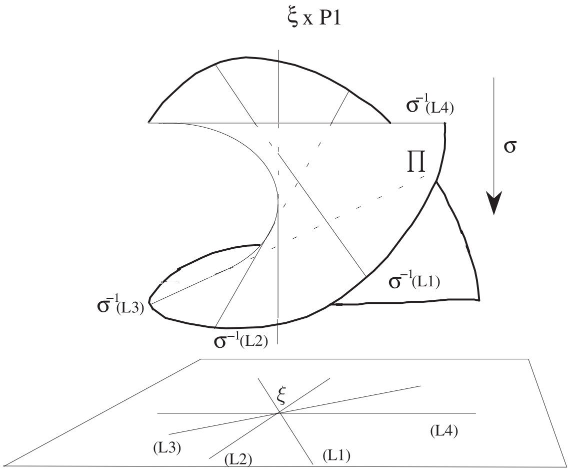

Figure 1. Illustration of the *blow-up* σ^{-1} of the point ξ and the respective images of some lines lying through the point ξ.

is a regular mapping onto L, it sends L to a curve in $\prod$ which meets $\xi \times P^{n-1}\mathbb{K}$ at the point $(\xi; \alpha_1 : \ldots : 1 : \ldots : \alpha_n)$ (see figure 1). Thus $\sigma^{-1}(n)$ is not regular at ξ since its value depends on the direction with which one tends to ξ. This remark is of crucial importance since it is at the basis of the concept of singularity confinement.

There exists an integer m such that, given any rational mapping F of $\mathbb{C}^n$, this transformation, once embedded in $\mathbb{PC}^m$ may be expressed in terms of homogeneous polynomials of the same order. The degree of this mapping F, denoted by $\deg(F)$ is the order of this polynomial. Hence, we have the property that given two mappings ϕ_1 and ϕ_2, one has

$$\deg(\phi_1 \circ \phi_2) \leqslant \deg(\phi_1) + \deg(\phi_2) \tag{2.5}$$

$(\phi_1, \phi_2) \in \mathbb{PC}^m[x_1, \ldots, x_{n+1}]$. This remark allows us to define an inner composition law $\times$ such that $\forall(\phi_1, \phi_2) \in \mathbb{C}^{m+1}[x_1, \ldots, x_{n+1}]$:

$$\phi_1 \circ \phi_2 = m[\phi_1, \phi_2](\phi_1 \times \phi_2) \tag{2.6}$$

where $m[\phi_1, \phi_2]$ is a homogenous polynomial common to each component of $\phi_1 \times \phi_2$. Let $\phi^{[n]}$ be the n-fold product: $\phi^{[n]} = \underbrace{\phi \times \cdots \times \phi}_{n\ \text{times}}$ and one sets $d_n = \deg(\phi^{[n]})$. The *algebraic entropy* of the mapping ϕ, $h_{al}(\phi)$ is defined as the limit

$$h_{al}(\phi) = \lim_{n\to\infty} \frac{1}{n} \ln d_n. \tag{2.7}$$

One has automatically $h_{al}(\phi_1 \circ \phi_2) \leqslant h_{al}(\phi_1) + h_{al}(\phi_2)$ and h_{al} is a birational invariant. Surely enough, let ψ be a birational transformation and let $\phi' = \psi^{-1} \circ \phi \circ \psi$.

Then we have $d_n(\phi) \leqslant d_n(\phi') = \deg(\psi^{-1} \circ \phi^{[n]} \circ \psi) \leqslant \deg(\psi)^2 d_n(\phi)$. By taking this expression and going to the limit $n \to \infty$, we establish the result

$$h_{al}(\phi') = h_{al}(\phi).$$

The algebraic entropy can have intuitive interpretations. For instance, the interpretation of Arnol'd is as follows: consider a straight line, its successive transforms will intersect an arbitrary line a number of times, and this number is equal to h_{al}. It is the so-called Arnol'd complexity [2].

2.2. *Factorization scheme*

Singular points of birational transformation in projective space $\mathbb{PC}^n$ are straight lines of $\mathbb{C}^{n+1}$ which are mapped to the origin $[0, \ldots, 0]$. An algebraic geometry theorem shows that this set is of codimension at least equal to two. Let ϕ be a birational transformation, ϕ_h its homogenized version in $\mathbb{PC}^m$ and ψ_h its homogeneous inverse defined by $\psi \circ \phi = K_\psi \mathbb{I}d$ and $\phi \circ \psi = K_\phi \mathbb{I}d$, where K_ψ and K_ϕ are polynomials of degree $\deg(\phi_h) \cdot \deg(\psi_h) - 1$. The zeros of K_ϕ form an algebraic set $Z(K_\phi)$; they play an important role in the understanding of the factorization of $\phi^{[n]}$. Viallet and Bellon [3] in their paper *Algebraic entropy* have proved that for any birational mapping ϕ_2 of $\mathbb{PC}^n$, such that $\phi_1 \circ \phi_2 = m(\phi_1, \phi_2)(\phi_1 \times \phi_2)$, $m(\phi_1, \phi_2)$ is a product of certain factors of K_{ϕ_1} and K_{ϕ_2}. Applying this property to $\phi_2 = \phi_1^{[n]}$ we conclude that only factors of K_{ϕ_1} can factorize!

As an example [3] let us consider the Hénon transformation. This mapping of $\mathbb{C}^2$ has no singularity, embedded in the projective space $\mathbb{PC}^2$:

$$H : \begin{cases} \mathbb{PC}^2 \longrightarrow \mathbb{PC}^2 \\ \begin{bmatrix} x \\ y \\ t \end{bmatrix} \longmapsto \begin{bmatrix} t^2 + ty - ax^2 \\ bxt \\ t^2 \end{bmatrix} \end{cases} \tag{2.8}$$

where a and b are parameters. Here $K_H = 1$. Thus H has no singularities. Consequently, the iterated transforms cannot have arbitrary factorizations $d_n(H) = 2^n \Rightarrow h_{al}(H) = \ln(2)$. We can easily generalize this result to all polynomial transformations for which the algebraic entropy reaches its maximum value, equal to $\deg(H)$.

2.3. *Geometric interpretation*

Let us consider the successive iterations of ϕ:

$$\begin{aligned} &\phi \circ \phi = \phi \times \phi = \phi^{[2]} \\ &\phi \circ \phi^{[2]} = \phi \times \phi^{[2]} = \phi^{[3]} \\ &\vdots \\ &\phi \circ \phi^{[k-1]} = \phi \times \phi^{[k-1]} = \phi^{[k]} \\ &\phi \circ \phi^{[k]} = M_\phi \phi \times \phi^{[k]} \end{aligned} \tag{2.9}$$

such that the first factorization occurs at the $(k+1)^{\text{th}}$ iteration, M_ϕ is a product of factors of K_ϕ. The last equation means that the set $Z(M_\phi)$ is mapped onto a manifold with codimension >1 by ϕ. If this codimension turns out to be one, we say that we have a process of *singularity confinement* [14, 15]. This corresponds to the maximal factorization. Otherwise this codimension is larger than one. We shall illustrate these two cases by some examples, in particular by the discrete Painlevé transformation.

2.4. *Conjecture*

It can be easily proved that singularities of birational transformations give rise to factorizations and consequently lower the value of algebraic entropy. In some extreme cases, we may even have a polynomial growth of degrees of the iterated homogenous transformations, i.e. $h_{al} = 0$. This suggests the following conjecture: *If ϕ is a birational mapping such that its algebraic entropy $h_{al}(\phi)$ is equal to zero then ϕ has an invariant which is not necessarily an algebraic invariant.*

2.5. An example of computation of algebraic entropy

Let us consider a birational transformation ϕ which admits $\{x = 0\}$ as a singular manifold:

$$\phi : \begin{cases} \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} y + \dfrac{a}{x^m} \\ x \end{bmatrix} \end{cases} \tag{2.10}$$

where $m \in \mathbb{N} \setminus \{0\}$ and $a \neq 0$ is a parameter. For $a = 0$, ϕ is simply a permutation. Embedded in $\mathbb{PC}^2$, ϕ gives ϕ_h:

$$\phi_h : \begin{cases} \mathbb{PC}^2 \longrightarrow \mathbb{PC}^2 \\ \begin{bmatrix} x \\ y \\ t \end{bmatrix} \longmapsto \begin{bmatrix} yx^m + at^{m+1} \\ x^{m+1} \\ x^m t \end{bmatrix}. \end{cases} \tag{2.11}$$

The homogenous inverse is

$$\phi_h : \begin{cases} \mathbb{PC}^2 \longrightarrow \mathbb{PC}^2 \\ \begin{bmatrix} x \\ y \\ t \end{bmatrix} \longmapsto \begin{bmatrix} y^{m+1} \\ xy^m - at^{m+1} \\ y^m t \end{bmatrix} \end{cases} \tag{2.12}$$

such that $\phi_h \circ \phi_h^{-1} = y^{2(m+1)-1}\mathbb{I}d$ and $\phi_h^{-1} \circ \phi_h = x^{2(m+1)-1}\mathbb{I}d$. Thus the transformations $\phi \circ \phi^{[n]}$ will only give factors which are powers of x. Let us now analyse the singularities in projective space near the line $[0, u, 1]$, $u \in \mathbb{C}$, i.e. on lines such as $[\epsilon, u, 1]$ where ϵ is small. Obviously, the line $[0, u, 1]$ is mapped by ϕ^3 onto $[0, 0, 0]$. However, in $\mathbb{PC}^2$ we have

$$\begin{aligned} \begin{bmatrix} \epsilon \\ u \\ 1 \end{bmatrix} \underset{\phi_h}{\longrightarrow} \begin{bmatrix} a + u\epsilon^m \\ \epsilon^{m+1} \\ \epsilon^m \end{bmatrix} \underset{\phi_h}{\longrightarrow} \begin{bmatrix} a^m\epsilon^{m+1} + \mathrm{o}(\epsilon^{m+1}) \\ a^{m+1} + \mathrm{o}(\epsilon) \\ a^m\epsilon^m + \mathrm{o}(\epsilon^m) \end{bmatrix} \\ = \begin{bmatrix} \epsilon^{m+1} + \mathrm{o}(\epsilon^{m+1}) \\ a + \mathrm{o}(\epsilon) \\ \epsilon^m + \mathrm{o}(\epsilon^m) \end{bmatrix} \underset{\phi_h}{\longrightarrow} \begin{bmatrix} 2a + \mathrm{o}(\epsilon) \\ \epsilon^m + \mathrm{o}(\epsilon^{m+1}) \\ \epsilon^m + \mathrm{o}(\epsilon^{m+1}) \end{bmatrix}. \end{aligned} \tag{2.13}$$

Going to the limit $\epsilon \to 0$, by continuity of $\phi_h^{[3]}$ in $\mathbb{PC}^2$ we have $\phi_h^{[3]}[0, u, 1] = [1, 0, 0]$. Thus the factorization of $\epsilon^{m(m+1)}$ of $\phi_h^{[3]}$ has regularized partially the singularity since we obtain at the end, a manifold of codimension two: here this is a point, different from $[0, 0, 0]$. This illustrates the case for which incomplete *regularization* occurs: it does not turn out to be a line. We obtain the following factorization scheme:

$$\begin{aligned} &\phi \circ \phi = \phi^{[2]} \\ &\phi \circ \phi^{[2]} = x^{m(m+1)}\phi \times \phi^{[2]} = x^{m(m+1)}\phi^{[3]}. \end{aligned} \tag{2.14}$$

The following diagram shows that the confinement of singularities are only partial, since $\phi_h^{[3]}[0, 0, 1]$ is mapped onto a point of period 2. The straight line $[0, u, 1]$, $u \in \mathbb{C}$ is blown down to the point $\bullet$ which in turn is mapped to the point $[0, 0, 0]$ by ϕ_h and back to $\bullet$.

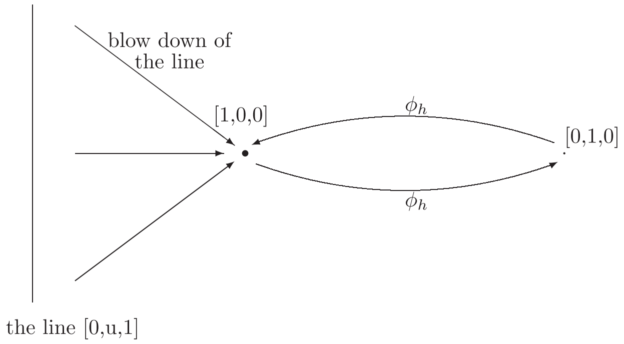

Figure 2. Illustration of the factorization scheme of ϕ.

Let us now compute the algebraic entropy. We have $\phi^{[2]^*}(x) = x^{m+1}x^{[2]}$, which means that the first component of $\phi^{[2]}$ factorizes x^{m+1}. But if one applies $\phi^{[2]^*}(x)$ to $\phi^*(x)$ then the factorization occurs: $x^{[1]^{m+1}}$ and consequently: $\phi^{[n]^*}(x) = x^{[n-2]^{m+1}}x^{[n]}$. Recall that $g_n = \deg(x^{[n]})$ and $d_n = \deg(\phi^{[n]})$, and the previous arguments translate into

$$d_n = g_{n-2}^{m+1} + g_n. \tag{2.15}$$

Making one more factorization step

$$\begin{aligned} &\phi \circ \phi^{[2]} = x^{m(m+1)}\phi^{[3]} \\ &\vdots \\ &\phi \circ \phi^{[n]} = x^{[n-2]^{m(m+1)}}\phi^{[n+1]} \end{aligned} \tag{2.16}$$

we have $d_n \deg(\phi) = m(m+1)g_{n-2} + d_{n+1}$ and $\deg(\phi) = m + 1$. Hence we have the system

$$\begin{aligned} d_{n+1} &= (m+1)(d_n - m g_{n-2}) \\ d_n &= (m+1)g_{n-2} + g_n \end{aligned}$$

from which one extracts $g_{n+2} = (m+1)(-g_{n+1} + g_n - g_{n-1})$. This linear recursion relation may be solved by using a generating function: $g(z) = \sum_{n\geqslant 0} g_n z^n$, and one finds for any m, an irreducible rational function:

$$g(z) = \frac{1 - z^2(m^2-1)}{1-(m+1)(z^3 - z^2 + z)}. \tag{2.17}$$

Some calculations show that $h_{al} = -\ln|\lambda|$, where λ is the root of the denominator of g which has the smallest modulus, note that $1/\deg(\phi) \leqslant |\lambda| \leqslant 1$ which implies that $h_{al} = (\phi) \geqslant 0$. For this example, the denominator is a polynomial of the order of three, having at least one root with modulus smaller that 1, implying that $h_{al} > 0$. For $m = 1$, ϕ is trivially integrable. Indeed, setting $U = xy$ and $V = y$, we have $\phi^*(U) = U + a$ and $\phi^*(V) = U/V$. This trivial but appropriate change of variables will decouple the system:

$$U_n \equiv \phi^{[n]^*}(U) = U_0 + na \tag{2.18}$$

$$V_n \equiv \phi^{[n]^*}(V) = \begin{cases} \dfrac{U_{n-1}U_{n-3}\ldots U_1}{U_{n-2}U_{n-4}\ldots U_0}\dfrac{V_0}{1} & n \text{ even} \\[2ex] \dfrac{U_{n-1}U_{n-3}\ldots U_0}{U_{n-2}U_{n-4}\ldots U_1}\dfrac{1}{V_0} & n \text{ odd.} \end{cases} \tag{2.19}$$

By considering ϕ^2, we have without loss of generality:

$$V_n = \frac{\Gamma\left(\frac{U_0}{2a}\right)}{\Gamma\left(\frac{3}{2}+\frac{U_0}{2a}\right)} \frac{\Gamma\left(n+\frac{1}{2}+\frac{U_0}{2a}\right)}{\Gamma\left(n-1+\frac{U_0}{2a}\right)}. \tag{2.20}$$

And substituting n by $\frac{U_n-U_0}{a}$, we obtain

$$V_n = \frac{\Gamma\left(\frac{U_0}{2a}\right)}{\Gamma\left(\frac{3}{2}+\frac{U_0}{2a}\right)} \frac{\Gamma\left(\frac{U_n}{a}-\frac{U_0}{2a}+\frac{1}{2}\right)}{\Gamma\left(\frac{U_n}{a}-\frac{U_0}{2a}+1\right)} V_0. \tag{2.21}$$

The following quantity F is a conserved quantity:

$$F(x, y, x_0, y_0) = x + x_0 - \frac{\Gamma\left(\frac{y_0}{2a}\right)}{\Gamma\left(\frac{3}{2}+\frac{y_0}{2a}\right)} \frac{\Gamma\left(\frac{y}{a}-\frac{y_0}{2a}+\frac{1}{2}\right)}{\Gamma\left(\frac{y}{a}-\frac{y_0}{2a}+1\right)}. \tag{2.22}$$

And it is obviously a *non-algebraic* invariant; furthermore, this is the *minimal* one by construction. The previous computation shows clearly its limitations, one gets only an upper bound for the algebraic entropy. In this case there only exists a partial confinement of the singularity. Consequently, there could still be additional factorizations which markedly affect the value of the algebraic entropy. Here in this example, the root of smallest modulus is equal to 0.64, yielding $h_{al} \simeq 0.43$ and from the previous considerations one should have $h_{al} = 0$.

Moreover, the analysis of singularities is only efficient in the case of an irreducible manifold. For generic cases, this method only provides a rough estimate, since irreducible manifolds mix themselves in the singularity confinement analysis.

3. The discrete Painlevé I transformation

This is a transformation for which there is a *complete regularization* of the singularities. Let σ_N be a cyclic mapping of $\mathbb{C}^N$ of the order of N perturbed uniquely by the addition of a pole $\frac{1}{x}$. We shall see why the discrete Painlevé I transformation plays an important role within this class of transforms.

$$\sigma_N : \begin{cases} \mathbb{C}^N \longrightarrow \mathbb{C}^N \\ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} \longmapsto \begin{bmatrix} x_2 + \dfrac{a_1}{x_1} \\ x_3 + \dfrac{a_3}{x_1} \\ \vdots \\ x_N + \dfrac{a_N}{x_1} \\ x_1 \end{bmatrix} \end{cases} \tag{3.1}$$

or embedded in the projective space:

$$\sigma_N^h : \begin{cases} \mathbb{PC}^N \longrightarrow \mathbb{PC}^N \\ \begin{bmatrix} x_i \\ x_N \\ t \end{bmatrix} \longmapsto \begin{bmatrix} x_{i+1}x_1 + a_i t^2 \\ x_1^2 \\ x_1 t \end{bmatrix} \end{cases} \tag{3.2}$$

for $1 \leqslant i < N - 1$.

Only powers of x_1 can be factorized from iteration of σ_N^h. Finally, $(\sigma_N^h)^N$ is the smallest power of σ_N which maps the manifold of codimension one $(\sigma_N^h)^N$ $([0, u_1, \ldots, u_N])$ onto $[0, \ldots, 0]$. Let us consider the neighbourhood of this hyperplane. Then we have the sequence of iterations of σ_N^h:

$$\begin{bmatrix} \epsilon \\ u_2 \\ \vdots \\ u_{N-3} \\ u_{N-2} \\ u_{N-1} \\ u_N \end{bmatrix} \longmapsto \begin{bmatrix} a_1 + \epsilon u_2 \\ a_2 + \epsilon u_3 \\ \vdots \\ u_{N-2} + \epsilon u_{N-1} \\ u_{N-1} + \epsilon u_N \\ \epsilon^2 \\ \epsilon \end{bmatrix} \longmapsto \begin{bmatrix} a_2 + \epsilon P_1 \\ a_3 + \epsilon P_2 \\ \vdots \\ u_{N-1} + \epsilon P_{N-1} \\ \epsilon^2 \left(1 + \dfrac{a_{N-1}}{a_1}\right) \\ a_1 + \epsilon P_N \\ \epsilon + \epsilon^2 P_{N+1} \end{bmatrix} \cdots$$

$$\longmapsto \begin{bmatrix} \epsilon^2(1+D) + \epsilon^3 P_1 \\ a_1 + \epsilon P_2 \\ \vdots \\ a_{N-3} + \epsilon P_{N-3} \\ a_{N-2} + \epsilon P_{N-2} \\ a_{N-1} + \epsilon P_N \\ \epsilon + \epsilon^2 P_{N+1} \end{bmatrix} \longmapsto \begin{bmatrix} \epsilon^2 a_1(1+D) + \epsilon^3 P_1 \\ \epsilon^2 a_3(1+D) + \epsilon^3 P_2 \\ \vdots \\ \epsilon^2 a_{N-2}(1+D) + \epsilon^3 P_{N-2} \\ \epsilon^2 a_{N-1}(1+D) + \epsilon^3 P_{N-1} \\ \epsilon^4 D^2 + \epsilon^5 P_N \\ \epsilon^2 D + \epsilon^4 P_{N+1} \end{bmatrix} \tag{3.3}$$

with

$$D = \frac{a_{N-1}}{a_1} + \cdots + \frac{a_2}{a_{N-2}} + \frac{a_1}{a_{N-1}}$$

$a_i \neq 0$ and P_i are generic polynomials in ϵ, $1 \leqslant i \leqslant N-1$. We note that a unique factorization of ϵ^2 can occur such that the system stays partially regularized. Following the same procedure as before, $g_n = \deg(x^{[n]})$ and $d_n = \deg(\sigma^{[n]})$, we may establish the equations, for $N \geqslant 1$:

$$\begin{aligned} d_n &= g_n + 2g_{n-N} \\ d_{n+1} &= 2d_n - 2g_{n-N} \end{aligned} \tag{3.4}$$

and derive the recursive relation for g_n:

$$g_n - 2g_{n-1} + 2g_{n-N} - 2g_{n-N-1} = 0. \tag{3.5}$$

One may show that the generating function of the g_n: $g(z) = \sum_{n\in\mathbb{N}} g_n z^n$ is, for a given $N \geqslant 1$:

$$g(z) = \frac{1}{1 - 2z + 2z^N - 2z^{N+1}}. \tag{3.6}$$

The smallest root of this polynomial has a linear approximation, in a neighbourhood of $z_0 = \frac{1}{2}$: $z \simeq z_0 + \left(\frac{z_0}{2}\right)^{N+1}$. Hence for a transformation of large order, the regularization is clearly insufficient and

$$h_{al}(\sigma_N) \xrightarrow[N\to\infty]{} \ln(\deg(\sigma_N)). \tag{3.7}$$

One could also discuss the case with $a_i = 0$, $1 \leqslant i < N-1$. This leads to a smallest algebraic entropy since the factorization of ϵ^2 occurs earlier. Another interesting case is the one for which $D = -1$. There still exists an additional factorization, corresponding to the self-regularization of the system; this means that we recover a manifold of codimension one after $N + 1$ iterations, and end up with the equation

$$g_n - 2g_{n+1-1} + 2g_{n+N} - g_{n+N+1} = 0 \tag{3.8}$$

for which the generating function is

$$g(z) = \frac{z^N - 1}{1 - 2z + 2z^N - z^{N+1}}. \tag{3.9}$$

So for $N = 3$,

$$g(z) = -\frac{1 + z + z^2}{(1 + z)(1 - z)^2}$$

hence $h_{al}(\sigma^3) = 0$. For generic $N > 3$ the solution is expanded to first order and there exists a root z_0 of modulus smaller than 1: $z_0 \simeq 1 - 2/N$. Thus $h_{al}(\sigma^N)_{N>3} > 0$. This shows that if the blow up does not occur sufficiently earlier, factorizations cannot lower the growth rate to zero, according to the dimension of the space. The discrete Painlevé transformation ($N = 3$ and $D = 1$) appears to be the only one to admit a zero algebraic entropy. It may be formulated as a two-variable transformation:

$$\sigma_3 : \begin{cases} \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} y + \dfrac{a}{x} \\ z + \dfrac{b}{x} \\ x \end{bmatrix} \end{cases} \tag{3.10}$$

with $D = 1 + \frac{a}{b} + \frac{b}{a}$ and the equation $D = -1$ determines a manifold of parameters with maximum factorizations: $a + b = 0$. Thus in the invariant plane $\{x + y + z = \text{constant}\}$. This yields a family of transformations depending on two parameters $(b, c) \in \mathbb{C}^2$:

$$\Phi : \begin{cases} \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} -x - y + b + \dfrac{c}{x} \\ x \end{bmatrix}. \end{cases} \tag{3.11}$$

There exists a non-autonomous version of this transformation which is called the discrete Painlevé I transformation:

$$\Phi_n : \begin{cases} \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} -x - y + b + \dfrac{c_n}{x} \\ x \end{bmatrix} \end{cases} \tag{3.12}$$

where $c_n = \alpha + \beta n + (-1)^n \gamma$ with α, β and γ being constants. This result comes from the demand that the system be self-regularized after the fourth iteration which forces c_n to fulfil the relation [14]

$$c_{n+1} - c_n = c_{n-1} - c_{n-2}. \tag{3.13}$$

3.1. The elliptic parametrization of the autonomous discrete Painlevé I transformation

As we have just seen, this transformation is obtained by perturbation of a cyclic transformation by adding poles in one variable. Hence we impose the resolution of singularities which gives a zero algebraic entropy. This property leads to the existence of an algebraic invariant (conserved) quantity:

$$\Delta : \begin{cases} \mathbb{C}^2 \longrightarrow \mathbb{C} \\ (x, y) \longmapsto (x + y - b)(xy - c). \end{cases} \tag{3.14}$$

Φ has numerous properties. It preserves areas and orientation since $\det(J_{\Phi_0}) = 1$. As a two-variable transformation, this symplectic mapping is a birational one such that the inverse is conjugate to the original transformation by an involution: $I : (x, y) \longmapsto (y, x)$. On the elliptic curves $\Delta(x, y) =$ constant which fill up the space by foliations, the transformation is simply a translation in the uniformizing variable. In fact, Δ may be viewed as the first integral of a Hamiltonian transformation.

Here we consider the dP-I, written in a recursive way:

$$\begin{aligned} x_{n+1} &= -x_n - y_n + b + \frac{3c}{x_n} \\ y_{n+1} &= x_n \end{aligned} \tag{3.15}$$

where n is an integer. This mapping has a known invariant given by

$$(x_n + y_n - b)(x_n y_n - 3c) = e \tag{3.16}$$

where e depend on the initial condition $e = (x_0 + y_0 - b)(x_0 y_0 - 3c)$, b and c are the real parameters of the transformation. We show now that a complete parametrization of this algebraic curve of third order may be obtained in terms of Weierstrass elliptic functions $\mathcal{P}(z; g_2, g_3)$ and $\mathcal{P}'(z; g_2, g_3)$ [4].

3.1.1. Parametrization. Introducing a parameter t through the relations

$$\begin{aligned} (x + y - b) &= \frac{e}{t} \\ (xy - 3c) &= t \end{aligned} \tag{3.17}$$

we establish the following representation of x and y in terms of elliptic Weierstrass functions:

$$\begin{aligned} x(\xi) &= \frac{1}{2}\left(b + \frac{\mathcal{P}'(v; g_2, g_3) - \mathcal{P}'(\xi; g_2, g_3)}{\mathcal{P}(v; g_2, g_3) - \mathcal{P}(\xi; g_2, g_3)}\right) \\ y(\xi) &= \frac{1}{2}\left(b + \frac{\mathcal{P}'(v; g_2, g_3) + \mathcal{P}'(\xi; g_2, g_3)}{\mathcal{P}(v; g_2, g_3) - \mathcal{P}(\xi; g_2, g_3)}\right) \end{aligned} \tag{3.18}$$

where ξ, g_2 and g_3 are given in appendix B. The dP-I mapping is now represented by a simple addition $\xi \mapsto \xi + v$, where v fulfils the relations

$$\begin{aligned} \mathcal{P}(v; g_2, g_3) &= \tfrac{1}{12}b^2 - c \\ \mathcal{P}'(v; g_2, g_3) &= e. \end{aligned} \tag{3.19}$$

3.2. The periodic points of the autonomous dP-I

As announced, the transformation is a left product:

$$\Delta : \begin{cases} \mathbb{C}^2 \longrightarrow \mathbb{C} \\ (\Delta, \theta) \longmapsto (\Delta, \theta + \tau(\Delta)) \end{cases} \tag{3.20}$$

with $\tau(\Delta)$ determined by the equation $\mathcal{P}(\tau(\Delta), g_2, g_3) = -z/3$. In this formulation and restricted to $\mathbb{R}$, the Hamiltonian may be simply expressed as [21]

$$H(\Delta) = \int_0^\Delta \tau(u)\, \mathrm{d}u. \tag{3.21}$$

A more explicit expression of $H(\Delta)$ is difficult to obtain because of the difficulty in inverting the elliptic Weierstrass function. Periodic points are points resulting from the congruency of the shift τ with the periods of elliptic curves ω_1 and ω_2. The values of ω_1 and ω_2 may be obtained [17]

$$\begin{aligned} \omega_1 &= \frac{1}{12\sqrt{2}(1-e^2)}\left[7F\left(-\tfrac{1}{6}, \tfrac{7}{6}, 2, e^2\right) + 5(1-2e^2)F\left(\tfrac{1}{6}, \tfrac{7}{6}, 2, e^2\right)\right] \\ \omega_2 &= \frac{1}{12\sqrt{2}(1-e^2)}\left[5F\left(\tfrac{1}{6}, \tfrac{7}{6}, 2, e^2\right) + 7(1-2e^2)F\left(-\tfrac{1}{6}, \tfrac{7}{6}, 2, e^2\right)\right] \end{aligned} \tag{3.22}$$

with

$$e = \frac{\Delta}{4\frac{z}{3}\sqrt{\frac{z}{3}}}$$

and F is the usual hypergeometric function, and the appearance of e is only due to the homogeneity property of the Weierstrassian functions. A point Q of the elliptic curve is a N-periodic point: $\Phi^{(N)}(Q) = Q$ if and only if

$$N\tau(\Delta(Q)) \equiv 0 \quad \mathrm{mod}(\omega_1, \omega_2). \tag{3.23}$$

Using addition formulae for the Weierstrass functions, we conclude that if a point Q is N-periodic then all the curves to which it belongs are made of N-periodic points. Thus the conditions $\Phi^{(N)}(x, y)$ may factorize polynomials of this kind:

$$\prod_{i=1}^{d_N}\left[\Delta(x, y) - a_i^{[N]}\right] \tag{3.24}$$

where $\{\Delta(x, y) = a_i^{[N]}\}$, $i = 1, \ldots, d_N$ are N-periodic manifolds, and d_n does increase in a polynomial way since $h_{al}(\Phi) = 0$. $a_i^{[N]}$ are solutions of polynomials whose degrees increases in a polynomial way with N since they are determined by the equations $\mathcal{P}(\tau, g_2, g_2) = 0$ or $\mathcal{P}'(\tau, g_2, g_2) = 0$. Using additional relations of the Weierstrass function [8] we conclude that the number of periodic manifolds is denumerable.

4. Multiplicative perturbation of the dP-I transformation

Let ϕ_λ be the birational transformation

$$\phi_\lambda : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} -x - y + z/x \\ x \\ \lambda z \end{bmatrix}. \end{cases} \tag{4.1}$$

Its inverse is

$$\phi_\lambda^{-1} : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} -x - y + \dfrac{z}{\lambda x} \\ x \\ \dfrac{z}{\lambda} \end{bmatrix}. \end{cases} \tag{4.2}$$

As we have seen, generically, this transformation does not self-regularize, except for discrete values of λ. Here we seek to establish the existence of an invariant by perturbative construction from a known transformation. This turns out to be a perturbation of the order of three in two variables.

4.1. Algebraic entropy by singularity confinement analysis

Consider the mapping F_n:

$$F_n : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ \begin{bmatrix} x \\ y \\ t \end{bmatrix} \longmapsto \begin{bmatrix} -x^2 - xy + c_n t^2 \\ x^2 \\ xt \end{bmatrix}. \end{cases} \tag{4.3}$$

By a singularity confinement analysis, we obtain the graph, for the line $[\epsilon, u, 1]$ close to the singular line $[0, u, 1]$.

We now show that self-regularization occurs if, $\forall (n, k) \in \mathbb{N}/\{0\}$:

$$(c_{3n+k} - c_{3n+k-1} - c_{3n+k-2}) + \cdots + (c_3 + k - c_2 + k - c_1 + k + c_k) = 0 \tag{4.4}$$

and only $F_{3n} \circ F_{3n-1} \circ F_{3n-2}$ will self-regularize. Suppose that the regularization has not yet occurred, so the image of the line $[\epsilon, u, 1]$ after the $3(n-1)^{\text{th}}$ iteration, in powers of ϵ, is

$$\begin{bmatrix} x \\ y \\ t \end{bmatrix} \xrightarrow[F_{3n-3}\circ\cdots\circ F_1 \circ F_0]{} \begin{bmatrix} M_n + \epsilon Q_1(u, \epsilon) \\ \epsilon^2 N_n + \epsilon^3 Q_2(u, \epsilon) \\ \epsilon P_n + \epsilon^2 Q_3(u, \epsilon) \end{bmatrix} \tag{4.5}$$

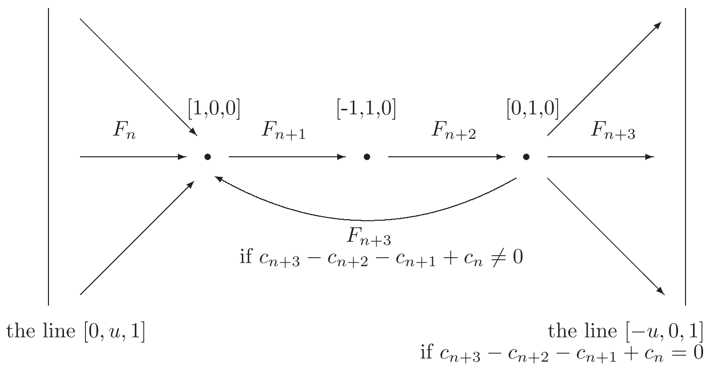

Figure 3. Factorization scheme of F_n.

where Q_1, Q_2 and Q_3 are generic polynomials in (u, ϵ); and it gives, through $F_{3n} \circ F_{3n-1} \circ F_{3n-2}$ (see figure 3):

$$\begin{bmatrix} -M_n^6 \epsilon^2 [-M_n N_n + (C_{3n+1} + C_{3n+2} - C_{3n+3}) P_n^2] + \mathrm{o}(\epsilon^2) \\ M_n^4 \epsilon^4 [-M_n N_n + C_{3n+1} P_{n-1}^2 + C_{3n+2} P_n^2]^2 + \mathrm{o}(\epsilon^4) \\ M_n^5 P_n \epsilon^3 [-M_n N_n + (C_{3n+1} + C_{3n+2}) P_n^2] + \mathrm{o}(\epsilon^3) \end{bmatrix} \tag{4.6}$$

with

$$\begin{aligned} C_0 &= -c_0 \\ C_1 &= -c_0 + c_1 \\ C_2 &= -c_0 + c_1 + c_2 \\ C_3 &= -c_0 + c_1 + c_2 - c_3 \\ C_4 &= -c_0 + c_1 + c_2 - c_3 + c_4 \ldots . \end{aligned} \tag{4.7}$$

One shows by recursion that

$$\begin{aligned} M_n &= -C_0^2 \ldots C_{3(n-2)}^2 C_{3(n-1)} \\ N_n &= C_2^2 \ldots C_{2+3k}^2 \ldots C_{2+3(n-2)}^2 \\ P_n &= -C_0 \cdot C_2 C_3 \ldots C_{2+3k} C_{3+3k} \ldots C_{2+3(n-3)} C_{3+3(n-3)} \cdot C_{2+3(n-2)}. \end{aligned} \tag{4.8}$$

Note that $M_n N_n = -P_n^2 C_{3(n-1)}$, such that there can be only regularization at the $(3n+1)^{\text{th}}$ step of the iteration, for $n \in \mathbb{N}$: $C_{3n} = 0$. Thus, in our case $C_n = \lambda^n z_0$, this would imply that

$$(\lambda^3 - \lambda^2 - \lambda) \sum_{k=0}^{n} \lambda^{3k} + 1 = 0. \tag{4.9}$$

Let us now evaluate the algebraic entropy for an arbitrary value of λ, i.e. in the case where no regularization occurs. It is easy to obtain the system of equations on the degree of the iterates as in [2], $\forall n \geqslant 4$:

$$\begin{aligned} d_{n+1} &= 2d_n - 2g_{n-3} \\ d_n &= g_n + 2g_{n-3}. \end{aligned} \tag{4.10}$$

One obtains the recursion relation

$$g_{n+4} - 2g_{n+3} + 2g_{n+1} - 2g_n = 0. \tag{4.11}$$

The generating function $g(z)$ has a denominator equal to $-2z^4 + 2z^3 - 2z + 1$. The value of the algebraic entropy is $h_{al}(\phi_\lambda) = -\ln z_0$, where z_0 is the smallest root of this polynomial: numerically $z_0 = 0.582$.

In the case where the regularization occurs at $m = 3n + 1$ step, i.e.

$$(\lambda^3 - \lambda^2 - \lambda) \sum_{k=0}^{n} \lambda^{3k} = -1$$

we have the following equations:

$$\begin{aligned} d_{n+m} &= 2d_{n+m-1} - 3g_{n+m-4} \\ d_{n+m-1} &= 2d_{n+m-2} - 3g_{n+m-5} \\ &\vdots \\ d_{n+1} &= 2d_n - 2g_{n-3} \end{aligned} \tag{4.12}$$

for $m \geqslant 4$ we also have $d_n = g_n + 2g_{n-3}$. Consequently, we have

$$d_{n+m} = g_{n+m} + 2g_{n+m-3} = 2^m d_n - 2^{m+1}\sum_{p=1}^{m-1} 2^{-p} g_{n-4+p} - 3g_{n+m-4}.$$

And, finally,

$$g_{n+m+3} + 2g_{n+m} + 3g_{n+m-1} + \sum_{p=2}^{m} 2^p g_{n+m-p} - 2^m g_{n+3} - 2^{m+1} g_n = 0. \qquad (4.13)$$

This yields a denominator of the generating function $g(z)$ equal to

$$\frac{(1-2z+2z^3-2z^4)(2^m z^m - 1)}{2z-1} + z^4. \qquad (4.14)$$

We give some numerical computation of the algebraic entropy of the 10 first steps of regularization $m = 3n+1$:

	smallest root	algebraic entropy
$n=1$	$z_0 = 1.0000$	$h_{al} = 0.0000$
$n=2$	$z_0 = 0.5979$	$h_{al} = 0.5142$
$n=3$	$z_0 = 0.5889$	$h_{al} = 0.5293$
$n=4$	$z_0 = 0.5858$	$h_{al} = 0.5346$
$n=5$	$z_0 = 0.5844$	$h_{al} = 0.5370$
$n=6$	$z_0 = 0.5836$	$h_{al} = 0.5384$
$n=7$	$z_0 = 0.5832$	$h_{al} = 0.5391$
$n=8$	$z_0 = 0.5829$	$h_{al} = 0.5396$
$n=9$	$z_0 = 0.5827$	$h_{al} = 0.5399$
$n=10$	$z_0 = 0.5826$	$h_{al} = 0.5400.$

(4.15)

To estimate the algebraic entropy we expand linearly the smallest root r of this polynomial around its asymptotic value (z_0 such that z_0 is the smallest root of $1 - 2z_0 + 2z_0^3 - 2z_0^4 = 0$), i.e. $z_0 \simeq 0.582$.

$$r \simeq z_0 + \frac{z_0^4(1-2z_0)}{2(2^m z_0^m - 1)(-1+3z_0^2-4z_0^3) + 2z_0^3(2z_0-1)}. \qquad (4.16)$$

We conclude that for $\lambda \neq 1$, $h_{al}(\phi_\lambda) > 0$ even when the regularization of the transformation occurs for $m > 1$. This is due to the fact that a *later* factorization will increase the algebraic entropy slightly. Nevertheless, some more factorizations may occur, altering the recursion relation for g_n and consequently the generating function. Some numerical studies have been made concerning this problem for $n = 2$ by Viallet and Hietarinta in [16]. Using algorithms, based on the first factorizations, we can reasonably guess the form of the generating function to be

$$g(z) = \frac{1+2z^3+2z^6}{(1-z)(1-z-z^2+z^3-z^4-z^5+z^6)}. \qquad (4.17)$$

Note that the algebraic entropy is still strictly positive.

4.2. Computation of the invariant

In this section, we propose to construct order by order a formal series depending on variables (x, y, t) which is invariant under ϕ_a. We will show that the series converges on an open set of $\mathbb{R}^3$. But this convergence is not a *sufficient* condition for global invariance under the transformation.

As has already been seen, it is equivalent to consider the transformation

$$\phi_a : \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \longmapsto \begin{bmatrix} y + \dfrac{at}{x} \\ z - \dfrac{at}{x} \\ x \\ \lambda t \end{bmatrix} \tag{4.18}$$

where a is a perturbation parameter and $\{x + y + z = \text{constant}\}$ is an invariant manifold. Let D_0 be the minimal polynomial invariant under ϕ_0^*: $\phi_0^*(D_0) = D_0$:

$$D_0(x, y, z) = \frac{1}{xyz}. \tag{4.19}$$

Note that at the start the choice of the initial invariant is arbitrary since any function of D_0 is also an invariant of ϕ_0. Hence if D_a is an invariant of ϕ_a, constructed from D_0 by perturbation, the arbitrariness of D_a is related to that of D_0. Now in homogeneous coordinates we have

$$\phi_a^h : \begin{bmatrix} x \\ y \\ z \\ t \\ v \end{bmatrix} \longmapsto \begin{bmatrix} yx + atv \\ zx - atv \\ x^2 \\ \lambda tx \\ xv \end{bmatrix} \tag{4.20}$$

with

$$D_0(x, y, z, v) = \frac{v^3}{xyz}.$$

Let suppose that D_a may be expanded in terms of a: $D_a = \sum_{n\in\mathbb{N}} a^n \Delta_n$ with the obvious property $\phi_a^{h*}(D_a) = D_a$. We will determine D_n order by order so that

$$\phi_a^{h*}\left(\sum_{n=0}^{N} a^n \Delta_n\right) = \sum_{n=0}^{N} a^n \Delta_n + \mathrm{o}(a^N). \tag{4.21}$$

We show by recursion that

$$D(x, y, z, t) = \sum_{N\in\mathbb{N}} a^N \Delta_N(x, y, t) \tag{4.22}$$

$$\Delta_N(x, y, t) = t^N \sum_{a+b+c=2N+3} \frac{\delta^N_{a,b,c}(\lambda)}{x^a y^b z^c} \tag{4.23}$$

$$|\delta^N_{a,b,c}(\lambda)| \leqslant \kappa \mu^N \tag{4.24}$$

where $\delta^N_{abc}(\lambda)$ depends only on λ. The summation is over integer indices. The invariant variety is the algebraic affine set generated by $D \in \mathbb{R}^4$ and the hyperplane $\{x + y + z = \text{constant}\}$.

We also estimate $\forall\lambda$, $\exists\mu > 0$ such that the coefficient $\delta^n_{abc}(\lambda) \leqslant \kappa\mu^n$, where κ is a positive constant.

As $D_a(x, y, z, t)$ is a converging series in an open set for any a, it sufficient to choose λ small enough so that the series still converges. Let Ω_λ be the open set in (x, y, z, t)-space such that the series converges. When $\Omega_\lambda = \mathbb{R}^4$, the quantity D_1 so constructed is clearly an invariant.

Consider the mapping

$$h_\gamma : \begin{cases} \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \\ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \longmapsto \begin{bmatrix} \sqrt{\gamma}x \\ \sqrt{\gamma}y \\ \sqrt{\gamma}z \\ \gamma t \end{bmatrix}. \end{cases} \tag{4.25}$$

It commutes with ϕ_λ: $\phi_\lambda \circ h_\gamma = h_\gamma \circ \phi_\lambda$. Thus one can extend the domain of definition of Ω_λ using h_γ: $\forall P \in \mathbb{R}^3$, $\exists\gamma$ such that $h_\gamma(P) \in \Omega_\lambda$ and $D \circ \phi_\lambda(h_\gamma(P)) = D(h_\gamma(P))$ so we have $(D \circ h_\gamma) \circ \phi_\lambda(P) = D \circ h_\gamma(P)$ and consequently $D \circ h_\gamma$ is an invariant defined in the neighbourhood of P.

Let (D_i, U_i) be a pair of sets such that $(U_i)_{i\in I}$ is a covering by open sets which can be contracted into Ω_λ, i.e.

$$\forall i \in I, \exists\gamma_i \quad \text{such that} \quad h_{\gamma_i}(U_i) \subset \Omega_\lambda \quad \text{and} \quad D_i = D \circ h_{\gamma_i}. \tag{4.26}$$

We have just shown that it is a local invariant. Since the family of scaling transformations $((h_\gamma)_{\gamma\in\mathbb{R}}$ commute with $\phi_a)$, one can verify that $(D_i)_{i\in I}$ match on the overlap, i.e. $\forall P \in \mathbb{R}^4$, $P' = \phi_a(P)$ and U_{i_p} as well as $U_{i_{P'}}$ open sets of the covering such that $P \in U_{i_p}$ and $P' \in U_{i_{P'}}$, we have, on the open set $U_{i_p} \cap \phi_a^{-1}(U_{i_{P'}})$ containing P, since ϕ_a is continuous:

$$D_{i_{P'}} = D(h^{-1}_{\gamma_{i_P}} \circ \phi_a \circ h_{\gamma_{i_P}}). \tag{4.27}$$

Nevertheless, there is a last point to check. We should show the existence of a subspace invariant in Ω_λ. In this case, the orbit of any point can be recast as curves defined in the subspace by conjugations of h. But this is difficult to prove. Knowing that no polynomial growth of the degree of the iterated transformation is observed for $m > 1$, it is quite possible that such domains do not exist.

We conclude that the existence of a formally invariant quantity is *not sufficient* to prove that the system is integrable: there must also be a reasonable domain of definition. This method, contrary to the Newton algorithm is not designed to construct a conjugation between the initial transformation and the perturbed one. It is mainly designed to construct the invariant of the transformation, directly by iteration.

5. The non-autonomous discrete Painlevé transformation

In this section, we discuss the integrability, or existence of a first integral, of the projection in the plane (x, y) of the transformation

$$F_\epsilon : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ \begin{bmatrix} x \\ y \\ t \end{bmatrix} \longmapsto \begin{bmatrix} -x - y + (z+\epsilon)/x \\ x \\ z + 2\epsilon \end{bmatrix}. \end{cases} \tag{5.1}$$

Note that F_ϵ is equivalent to the transformation: $[x, y, z] \longmapsto [-x-y+z/x, x, z+2\epsilon]$ through the conjugation $[x, y, z] \longmapsto [x, y, z+2\epsilon]$. z can be thought of as a time variable and goes to its fixed point ∞. From a physical point of view, only asymptotic solutions, with z large, are relevant. We also observe that F_ϵ is a left product of $\mathbb{R}^2 \ltimes \mathbb{R}$:

$$F_\epsilon : \begin{bmatrix} (x, y) \\ z \end{bmatrix} \longmapsto \begin{bmatrix} \tau_z(x, y) \\ \phi(z) \end{bmatrix} \tag{5.2}$$

with $\phi(z) = z + 2\epsilon$ and $\tau_z(x, y) = (-x - y + (z + \epsilon)/x)$. Let P_r be the projection $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $(x, y, z) \longmapsto (x, y)$. It is important to note that the integrability of $P_r \circ F_\epsilon$ in $\mathbb{R}^2$ is *not* equivalent to the integrability of F_ϵ.

Thus, even if F_ϵ is integrable we must add, to verify the integrability of the discrete Painlevé I transformation, the following condition: $\mathcal{C} \in \mathbb{R}^2$ is an invariant curve of $P \circ F_\epsilon$ iff $\forall M \in \mathcal{C} \Rightarrow P\,(Orb(M)) \subset \mathcal{C}$.

One may view the situation as an autonomous 'driven' system through a coupling constant z. Another possible interpretation is the following. Set f_z: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $[x, y] \longmapsto [-x - y + (z+\epsilon)/x, x]$, we recall that $f_z = P_r \circ F_{\epsilon=0}$ and we see that

$$P_r\left(F_\epsilon^{(n)}\right) = f_{z+(2n+1)\epsilon} \circ f_{z+(2n-1)\epsilon} \circ \cdots \circ f_{z+\epsilon}. \tag{5.3}$$

So the non-autonomous dP-I dynamics is just an ordered composition of a family of similar transformations. This point of view may look a little awkward at first, but as we shall see later on, it will appear perfectly legitimate.

We may also view F_ϵ as a perturbation of poles of transformation: $[x, y, z] \longmapsto [-x - y, x, z + 2\epsilon]$ where the projection in the plane (x, y) is a cyclic transformation of the order of three.

Finally, by examining the expression of F_ϵ we can decrease the dimensionality of the problem by setting

$$\begin{aligned} x(t + 2\epsilon) &= F_\epsilon[x(t), y(t)z(t)]_1 \\ y(t + 2\epsilon) &= F_\epsilon[x(t), y(t)z(t)]_2 \end{aligned} \tag{5.4}$$

where t is the time and $F_\epsilon[\cdot]_i$, $i = 1, 2$, are the first and second components. One obtains the equation of motion: $x(t)[x(t+\epsilon) + x(t) + x(t-\epsilon)] = t + \epsilon$. This alternative form of dP-I has been studied extensively by Joshi [18, 19].

Inspired by works on the resolution of continuous Painlevé equations using isomonodromy methods, she constructed an asymptotic solution for $t \to \infty$ under the form of a formal algebraic series:

$$x(t) = \sum_{n\in\mathbb{N}} a_n t^{\frac{1}{2}(1-n)}. \tag{5.5}$$

She found that this solution *diverges* and that the equation has no solution in the limit $t \to \infty$. We shall comment on this conclusion later in the paper.

This example of dP-I has the interest that it has a zero algebraic entropy. So if an invariant exists as a conserved quantity, it must be *transcendental*. This example suggests that the formulated conjecture may be extended to non-algebraic invariants. As we have already seen, the mappings f_z refer to the stopping of the Hamiltonian flow at instant 1, obtained by a simple combination of Weierstrass functions. However, the main difficulty comes from the change of coordinates generated by the evolution $z \longmapsto z + 2\epsilon$, which makes the behaviour of the invariant of f_z: $\Delta_z(x, y) = (x, y)(xy - z)$ unbounded. As a matter of fact one has

$$f^*_{z+2\epsilon}[\Delta_z](x, y) = \Delta_z(x, y)\left(1 - \frac{2\epsilon}{x(x + y)}\right). \tag{5.6}$$

Thus the two elliptic invariants g_2 and g_3 will not remain uncontrollable in the neighbourhood of $\{x = 0\}$ and $\{x + y = 0\}$.

5.1. Other obstruction to direct computation of the invariant

If one seeks to construct an invariant order by order in ϵ as in the non-autonomous case generated by the multiplicative shift $z \longmapsto \lambda z$, one meets an insurmountable problem: $\left(f^*_z - \mathbb{I}d^*\right)$ is not formally invertible in the space of rational functions of three variables:

$$\left(\mathbb{I}d^* - f^*_z\right) \neq \sum_{n\in\mathbb{N}} f^{(n)*}_z. \tag{5.7}$$

As in the case of multiplicative perturbation (see appendix C), it is imperative to construct the inverse of this operator. It is also difficult to prove the existence of some invariant curves using the KAM theorem for diffeomorphisms. However, F_ϵ has a large number of properties.

- F_ϵ is conjugate to its inverse in $\mathbb{C}^3$, i.e. $F^{-1}_\epsilon = \sigma^{-1} \circ F_\epsilon \circ \sigma$ with σ a mapping of $\mathbb{C}^3$: $[x, y, z] \longmapsto [\mathrm{i}y, \mathrm{i}x, -z]$ and $\sigma^4 = \mathbb{I}d$. The obvious disadvantage of this conjugation is that it is complex and consequently increases the dimensionality of the problem from three to six.
- F_ϵ conserves volume and orientation since $\det(J_{F_\epsilon}) = 1$. However, the applicability of the KAM theorem requires exact conservation of volume [6]. The presence of the pole $\frac{1}{x}$ is an obstruction to the application of this theorem. Exact conservation of the volume means existence of a 2-form ω such that $F^*\omega - \omega = \mathrm{d}\tau$, where τ is a 1-form. Formally, if we choose $\omega = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z$, the previous equation will satisfy the previous equation for $\tau = (x - \ln x)\,\mathrm{d}z - y\,\mathrm{d}z$. Such a solution reflects a singularity, due to the pole, through the term $\ln x$ which diverges softly at $x \to 0$. The presence of the pole does not seem to prevent the application of the KAM theorem. Exact conservation of the volume is a global constraint which imposes that the perturbed curve, whenever it exists, should be globally closed to the initial curve. One should thus get a closed curve upon application of the perturbation. This constraint helps us to control what is happening in the neighbourhood of ∞. In the case of autonomous dP-I the curve $\Delta_z(x, y) = (x + y)(xy - z) =$ constant goes through the point ∞, yielding a closed curve in the compactification of $\mathbb{R}^2$ by the projective space $\mathbb{PR}^2$.

A second basic remark: in the autonomous case, the pole has no effect on the integrability. As we have seen, $f^{(4)}_z : \mathbb{R}^2 \setminus \{x = 0\} \longmapsto \mathbb{R}^2 \setminus \{y = 0\}$ sends the singular line $\{x = 0\}$ onto $\{y = 0\}$, and conversely, since f_z is birational. As we have seen previously, from the projective space point of view, one can define by continuity:

$$f^{(4)}_z \begin{bmatrix} \epsilon \\ y \end{bmatrix} = \begin{bmatrix} y + \mathrm{o}(\epsilon) \\ -\epsilon \end{bmatrix} \tag{5.8}$$

and for, $\epsilon \to 0$, we have $\forall z$:

$$f^{(4)}_z \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}. \tag{5.9}$$

The process, called *singularity confinement*, amounts to finding a power of the transformation, here the fourth power of f_z, which regularizes the behaviour of orbits in the neighbourhood of infinity. Thus if x and y are close to each other and close to $\{x = 0\}$ they keep the same order by f_z^4, contrary to intuition. But the major feature of this crossing of ∞ is that f_z^4 conserves areas: $f_z^{4*}(\mathrm{d}y \wedge d(-x)) = \mathrm{d}x \wedge \mathrm{d}y$.

5.2. *What happens in the non-autonomous case?*

F_ϵ sends the singular locus $\{x = 0\}$ into $\{y = 0\}$ and F_ϵ can be extended by continuity to the neighbourhood of $\{x = 0\}$,

$$F_e^{(4)} \begin{bmatrix} \epsilon \\ y \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{y}{1 + 3e/z} + \mathrm{o}(\epsilon) \\ -\epsilon\left(1 + \dfrac{3e}{z}\right) + \mathrm{o}(\epsilon^2) \\ z + 2e \end{bmatrix}. \tag{5.10}$$

By taking the limit $\epsilon \to 0$ we obtain

$$F_e^{(4)} \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{y}{1 + 3e/z} \\ 0 \\ z + 2e \end{bmatrix}. \tag{5.11}$$

This justifies the duality between $\{x = 0\}$ and $\{y = 0\}$. We compute the variation of the areas of F_ϵ^4:

$$F_\epsilon^{4*}(\mathrm{d}x \wedge \mathrm{d}y) = \frac{\mathrm{d}y}{1 + 3\epsilon/z} \wedge \left(1 + \frac{3\epsilon}{z}\right) d(-x) = \mathrm{d}x \wedge \mathrm{d}y. \tag{5.12}$$

Thus two points near each other and close to the neighbourhood of the singularity remain close to each other under the action of $F_\epsilon^{(4)}$. As in the autonomous case, the passage near infinity is totally structured as shown by the expression of $F_\epsilon^{(4)}(e, y, z)$, and $F_\epsilon^{(4)}$ still conserves the areas. We also note that in the limit $z \to \infty$ we recover the same behaviour 'in crossing' as in the autonomous case.

A last point, recall that f_z has torsion. Indeed, the addition $\tau_{z,E}$ on the elliptic curve $\Delta_z = E$ is defined by

$$\begin{aligned} \mathcal{P}'(\tau, g_2, g_3) &= E \\ \mathcal{P}(\tau, g_2, g_3) &= 3z \end{aligned} \tag{5.13}$$

with $g_2 = \frac{4}{3}z$ and $g_3 = E^2 - \frac{8}{27}z^3$. Thus by continuity, $\mathcal{P} \circ \tau_{z,E}$ sweeps a continuous interval of values. The shift $\tau_{z,E}$ is difficult to evaluate due to the complicated inversion formula of the Weierstass function. We recall that the transformation F_0 has torsion if an infinite number of values of $\tau_{z,E}$ exist which are badly closed to a relative integer:

$$\exists\alpha, \exists\gamma > 0 \qquad (k_1, k_2) \in \mathbb{Z} \setminus \{0\} \times \mathbb{Z} \setminus \{0\} : \quad |\langle \tau_{z,E}, k \rangle| \geqslant \gamma |k|^{-\alpha} \tag{5.14}$$

with $\langle \tau, k \rangle = k_1 \operatorname{Re}(\tau) + k_2 \operatorname{Im}(\tau)$. Consequently, among the set of curves $\{\Delta_z(x, y) = E\}$, those with this Diophantine condition are dense in $\mathbb{R}^2$. These are the curves which are conserved partially under the perturbation. Finally, all the required conditions are met to apply the KAM theorem [6], except the exact conservation of areas which is equivalent to the control of the lack of regularity at infinity. But the confinement of singularities helps to re-establish this control by imposing volume conservation when crossing to infinity. The arguments presented here are, in fact, conditions under which the KAM theorem may be extended to birational mappings which enjoy the property of singularity confinement. Our remark cannot replace, however, a real proof, but it is also not a speculation since, in the autonomous case, it is the unique reason for integrability.

Suppose now that the KAM theorem is applicable. It would follow that among the elliptic curves $\{\Delta_z(x, y) = E\}$ which are invariant under F_z, some of those which are badly approximated by integers will survive the perturbation. Let ϵ be the perturbation parameter, then $P_r \circ F_\epsilon \underset{\epsilon\to 0}{\longrightarrow} f_z$, where we recall that P_r is the (x, y) projection. Let $\mathcal{C}_\epsilon$ be the set of curves which are invariant under the perturbation. In the language of the Lebesgue probability measure, we have $\mu(\mathcal{C}_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 1$. Thus as $\epsilon \to 0$ the curves tend to occupy the whole space and eventually fill it completely at $\epsilon = 0^-$. We then get a dense union of curves $\Delta_z(x, y) = E$ in the plane $\mathbb{R}^2$. Moreover, consider the invertible mapping:

$$h_\epsilon : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} \sqrt{\epsilon} x \\ \sqrt{\epsilon} x \\ \epsilon z \end{bmatrix}. \end{cases} \tag{5.15}$$

We note that $\forall\epsilon > 0$, F_1 and F_ϵ are conjugate via h_ϵ in $\mathbb{R}^3$. Consequently, $\forall\epsilon' > 0$ the projections $P_r \circ F_\epsilon$ and $P_r \circ F'_\epsilon$ also conjugate one another. So thanks to the previous observation $\forall\epsilon > 0$, $P_r \circ F_\epsilon$ admits a dense set of invariant curves. Extending the existence of a dense set of invariant curves to the whole space is a non-trivial process. The argument that at any point M there exists, due to the conservation of areas, an invariant closed curve, as close as possible to the orbit going through M and consequently is an invariant closed curve is not valid. This cannot be traced back to the projection P_r of the dynamics F_ϵ in three dimensions: let M and M' be such that $P_r(M) = P_r(M')$ then $P_r(Orb(M)) \neq P_r(orb(M'))$. The obstruction is due to the impossibility of trapping the orbit through M by two closed curves which are arbitrarily close to the orbit of M.

5.3. Proof that the invariant curves are non-algebraic

According to the KAM theorem, invariant curves are conjugate to initial curves. We must prove that this conjugate curve is *not* algebraic. Let ϕ be defined by $F_0 = \phi^{-1} \circ F_\epsilon \circ \phi$, where F_0: $[x, y, z] \longmapsto [f_z(x, y), z]$. The third component is

$$\phi[F_0|_x, F_0|_y, F_0|_z]_z = F_\epsilon[\phi|_x, \phi|_y, \phi|_z]_3 \tag{5.16}$$

applied to the point $[x, y, z]$:

$$\phi[f_z(x, y), z]_z = \phi[x, y, z]_z + 2\epsilon. \tag{5.17}$$

Now consider this equation for the set of periodic points of F_0. As seen before, there exists an infinite number of curves such that the corresponding addition τ is commensurate with the two periods of the curve: $\omega_1(z, \Delta_z = E)$ and $\omega_2(z, \Delta_z = E)$, i.e. these varieties are exclusively formed by periodic points. Let M in the plane (x, y) be such that $f_z^{(n)}(M) = M$:

$$\left(\phi \circ F_0^{(n)}[M, z]\right)_3 = \phi[f_z(M), z]_3 = \phi[M, z]_3 + 2n\epsilon. \tag{5.18}$$

The equality is only possible if $\phi[M, z]_z$ is infinite. So ϕ_z admits an infinite number of divisors: $\Delta_z - \Delta_z(M) = (x + y)(xy - z) - (x_M + y_M)(x_M y_M - z)$, i.e. one for each periodic invariant

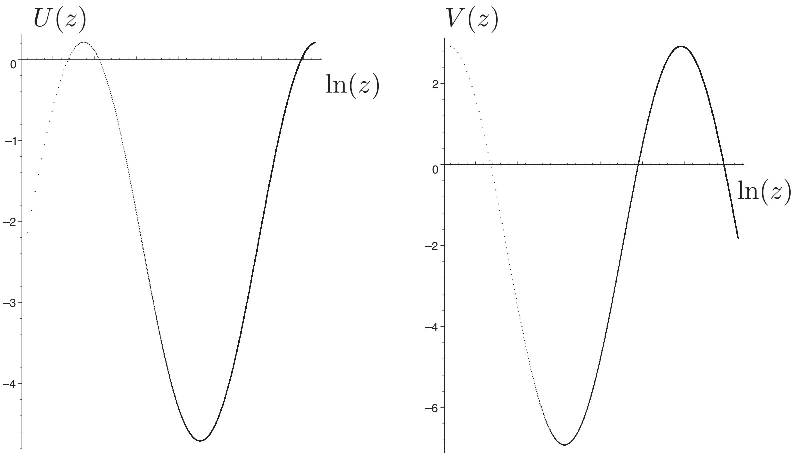

Figure 4. Computation of the coordinates U and V for a log time scale: $x = -1$, $y = 1$, $z = 1$ and $\epsilon = 2$.

variety for f_z. Consequently, ϕ is not algebraic: now let $\Omega_0 \circ F_0 = \Lambda_0$, where $\Lambda_0 = (\Delta_z, z)$ be the invariant for F_0 then $\Omega_\epsilon = \Omega_0 \circ \phi^{-1}$ is the invariant for F_ϵ. Thus Λ_ϵ is not algebraic. In fact, one should show that Λ_ϵ generates the ideal of invariants under F_ϵ, if Δ_0 generates the ideal of invariants of F_0. This would mean that any invariant under F_ϵ may be expressed in terms of Λ_ϵ and z. In other words, if we consider the mapping $G : \mathbb{R} \longmapsto \mathbb{R}$ then $(G \circ \Delta_\epsilon) \circ F_\epsilon = G \circ \Delta_\epsilon$ so $G \circ \Delta_\epsilon$ is also an invariant. Hence it remains to check that $\forall G : \mathbb{R} \longmapsto \mathbb{R}$, $G \circ \phi^{-1}$ is not a rational mapping.

6. Asymptotic solutions in the limit $z \to \infty$

In this section, we shall construct an asymptotic solution to the dP-I equation and discuss the result obtained recently by Joshi. To this end, we shall consider $F_\epsilon^{(4)}$ as it appears to be the most *regular* power of the transformation. Numerically, it appears that the projection of the orbits of $F_\epsilon^{(4)}$ *tends* to the curve $\{\Delta_z(x, y) = 0\}$. More precisely depending on the initial conditions and the value of ϵ the orbits under $F_\epsilon^{(4)}$ may *tend* to:

- the line $x + y = 0$
- or the image of this line under f_z: the hyperbola $f_z^*(x + y) = xy - z$.

One observes that the larger the parameter of perturbation ϵ, the more rapidly the dynamics converges: this is similar to the behaviour of forced differential systems. The stationary solutions are reached asymptotically. Because of the periodicity 4 of the variety $\{\Delta_z(x, y) = 0\}$, one goes by F_ϵ from the hyperbola to the straight line and conversely. For small ϵ, $P_r \circ F_\epsilon$ behave as f_z on $\{\Delta_z(x, y) = 0\}$, as shown by successive iterations:

$$\begin{bmatrix} x \\ -x \end{bmatrix} \underset{f_z}{\rightarrow} \begin{bmatrix} \dfrac{z}{x} \\ x \end{bmatrix} \underset{f_z}{\rightarrow} \begin{bmatrix} -\dfrac{z}{x} \\ \dfrac{z}{x} \end{bmatrix} \underset{f_z}{\rightarrow} \begin{bmatrix} -x \\ \dfrac{z}{x} \end{bmatrix} \underset{f_z}{\rightarrow} \begin{bmatrix} x \\ -x \end{bmatrix}. \tag{6.1}$$

Thus $\{\Delta_z(x, y) = 0\}$ appears to act as an attractor. We will now study $F_\epsilon^{(4)}x$ after its transient regime, when the orbit is closed to this variety. Numerical observations suggest that one may consider the orthogonal coordinates: $U = x + y$ and $V = x - y$. Even though U remains bounded, V diverges when $U \to 0$. As U and V do not decouple, a more appropriate coordinates choice would be

$$\begin{aligned} U &= (x+y)\sqrt{z} \\ V &= (x-y)(x+y). \end{aligned} \tag{6.2}$$

The graphs $(\ln z, U(z))$ and $(\ln z, V(z))$ (see figure 4) reveal that both U and V are periodic in $\ln z$ and may even be trigonometric in $\ln z$. In fact, numerical results suggest that

$$\begin{aligned} U(z) &= A\sin^2\theta(z) - a \\ V(z) &= 2B\sin\theta(z)\cos\theta(z) + b. \end{aligned} \tag{6.3}$$

Matching zeros of U and V in the two expressions yields

$$\begin{aligned} a &= A\sin^2\theta_0 \\ b &= 2B\sin\theta_0\cos\theta_0 \end{aligned} \tag{6.4}$$

so that

$$\begin{aligned} U(\theta) &= A\sin(\theta+\theta_0)\sin(\theta-\theta_0) \\ V(\theta) &= 2B\sin(\theta-\theta_0)\cos(\theta+\theta_0). \end{aligned} \tag{6.5}$$

We now show that this choice of variables is compatible with the drift toward $\{x + y = 0\}$. To that end, we will use the degeneracy of the Weierstrass function to the trigonometric function. Using the fact that the transformation reduces to an angle addition, in the limit $z \to \infty$, we deduce an asymptotic expansion of the variable $x(z)$ and $y(z)$. For details of the proof, see appendix D. Consequently, we obtain the expressions:

$$\begin{aligned} x(z) &= 2\sqrt{z}\cot\left[\phi - \frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon} + \theta_0\right] \\ y(z) &= -2\sqrt{z}\cot\left[\phi - \frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon} + \theta_0\right] \end{aligned} \tag{6.6}$$

where ϵ is the perturbation parameter and A and ϕ depend on initial conditions as is usual in the resolution of differential equations: $A_{x,y}$ and $\phi_{x,y}$.

Hence in the limit $z \to \infty$, the quantity $\Delta(x, y, z)$ is given by

$$\Delta(x,y,z) = -4A_{x,y}\sqrt{z}\frac{\sin\left[\phi_{x,y} - \frac{1}{8\sin(\theta_0(\phi_{x,y}))}\ln\frac{z}{2\epsilon} - \theta_0(\phi_{x,y})\right]}{\sin\left[\phi_{x,y} - \frac{1}{8\sin(\theta_0(\phi_{x,y}))}\ln\frac{z}{2\epsilon} + \theta_0(\phi_{x,y})\right]}. \tag{6.7}$$

Consequently, $\Delta(z)$ is generally speaking unbounded and it is not the working hypothesis of Joshi in [18]. It is the case if and only if $\theta_0 = 0 \pmod{\pi}$, one then recovers the autonomous case since

$$\epsilon = \frac{\sin(2\theta_0)}{A} + \mathrm{o}\left(\frac{1}{z}\right).$$

This result shows that some orbits tend to the curve $\Delta = 0$, which corresponds to the degeneracy of the Weierstrass functions into trigonometric functions. Numerical computations agree with this behaviour. One may now raise the question: *why does the system not converge towards*

$$\frac{\Delta}{z\sqrt{z}} = \pm\frac{2}{3\sqrt{3}}$$

for almost every z? This would correspond to another degeneracy with $e_2 = e_3$. Consider $f_z = h^{-1}_{\sqrt{z}} \circ f_1 \circ h_{\sqrt{z}}$, where $h_{\sqrt{z}} : [x, y] \longmapsto [\sqrt{z}x, \sqrt{z}y]$. Then the n^{th} iterate may be expressed as

$$\begin{aligned} f_{z+n\epsilon} &\circ f_{z+(n-1)\epsilon} \circ \cdots \circ f_{z+\epsilon} \circ f_z \\ &= h^{-1}_{\sqrt{z+n\epsilon}} \circ f_1 \circ h_{\sqrt{z+n\epsilon}} \circ h^{-1}_{\sqrt{z+(n-1)\epsilon}} \circ f_1 \circ \cdots \circ f_1 \circ h_{\sqrt{z}} \\ &= h^{-1}_{\sqrt{z+n\epsilon}} \circ f_1 \circ h_{\sqrt{(z+n\epsilon)/(z+(n-1)\epsilon)}} \circ \cdots \circ f_1 \circ h_{\sqrt{z}}. \end{aligned} \tag{6.8}$$

But when $n \to \infty$, we find

$$h_{\sqrt{(z+n\epsilon)/(z+(n-1)\epsilon)}} \simeq \mathbb{I}d + \mathrm{o}\left(\frac{1}{n}\right). \tag{6.9}$$

The dynamics F_ϵ is equivalent to the composition of f_1 and a family of similar mappings h_z, $z \in \mathbb{R}$, which tend to the identity mapping. There exists only one invariant manifold consistent with these two families of *symmetries* (f_1 and h_z): $\Delta = 0$. (The dynamics seeks to memorize the constraints as in a Hamiltonian system, where the result is a geodesic which minimizes the action and satisfies at best all the symmetries.)

We can define this asymptotic conserved quantity using coordinates U, V:

$$(V + \epsilon)^2 + (2U - \sqrt{A^2 - \epsilon^2})^2 \underset{z\to\infty}{\simeq} A^2 \tag{6.10}$$

where ϵ is the parameter of perturbation, A is determined by initial conditions U_0, V_0 such that $(V_0 + \epsilon)^2 + (2U_0 - \sqrt{A^2 - \epsilon^2})^2 \underset{z\to\infty}{\simeq} A^2$, as an integration constant. Here, necessarily $|\epsilon| \leqslant A$.

Remark. The solution obtained by Joshi in [19] in the form $x(t) = \sum_{n\in\mathbb{N}} a_n t^{(1-n)/2}$ can only be divergent in view of what we have obtained: the basis of solutions is too restrictive. One should have added a harmonic decomposition basis as is suggested by our computation. Moreover, the behaviour in $\ln t$ in the neighbourhood of infinity cannot be described by a Laurent series since $\ln t$ is not expandable in power series. We can go back to an expansion near zero, through the change of variable $t \to \frac{1}{t}$.

7. Conclusion

As presented in this study, the question of integrability of the dP-I mapping (equation) is certainly an interesting and challenging question. We have tried to give a partial answer which, we hope should inspire more work in order to give a definite answer. We have used all presently known techniques to deal with the problem: test of singularity confinement, computation of algebraic entropy, study of the growth of the complexity of the factorization, etc... As the invariant of the mapping in the non-autonomous case is certainly non-algebraic, its parametrization by known elliptic functions is excluded; one might think then of a parametrization perhaps by other transcendental functions such as Abelian functions. If this works, applications to statistical physics may be at hand since one would have new non-trivial models which are exactly solvable. There are topics which are not touched upon in this paper such as the possible role of the isomonodromic deformation in the determination of non-algebraic invariants and the connection of the spaces of initial conditions of Painlevé equations with the blow-ups of their singularities, leading to a classification according to the root systems of affine Weyl groups. Another direction of possible investigation would be the complete determination of asymptotic solutions of this equation under an explicit form which is workable for physical problems as well as its relations to other types of nonlinear discrete equations.

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Appendix A. Algebraic geometry

A.1. Some topics on Zariski topology

Consider the affine space $\mathbb{K}^n$ and the trivial ideals generated by elements 1 and 0, respectively, we have clearly $V(\{1\}) = \emptyset$ and $V(\{0\}) = \emptyset$. Hence the empty set and the whole space are algebraic affines sets. Let $(S_i)_{i\in I}$ be a family of subsets of $\mathbb{K}^n$. From the definition of algebraic sets we have

$$\bigcap_{i\in I} V(\langle S_i\rangle) = V\left(\left\langle \bigcap_{i\in I} S_i\right\rangle\right) \tag{A.1}$$

$\langle S\rangle$ denotes the ideal generated by S. Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals. We have $\mathcal{I}\mathcal{J} \subset \mathcal{I}, \mathcal{J}$ hence $V(\mathcal{I}) \cap V(\mathcal{J}) \subset V(\mathcal{I}\mathcal{J})$. Conversely, let $x \in V(\mathcal{I}\mathcal{J})$ and suppose that $x \notin V(\mathcal{I})$. There exists $P \in \mathcal{I}$ such that $P(x) \neq 0$. Then if $Q \in \mathcal{J}$ and $Q(x) = 0$ one has $PQ \in \mathcal{I}\mathcal{J}$ and consequently $(PQ)(x) = 0$. A similar reasoning shows that

$$V(\mathcal{I}) \cup V(\mathcal{J}) = V(\mathcal{I} \cap \mathcal{J}). \tag{A.2}$$

Thus the finite union of algebraic sets is an algebraic set. This completes the verification that these sets are closed sets for Zariski topology. This topology is intuitively very different from usual topologies: open sets are very 'large' so that two open sets in $\mathbb{K}^n$ always have a non-empty intersection.

A.2 Morphisms of algebraic sets

After giving the definition of objects in algebraic geometry, it remains to define mappings for these objects to turn them into categories, the *morphisms*. Let X be an affine variety of dimension n and let $R \in X$. A mapping $f : X \to \mathbb{K}$ is said to be regular in R if there exists an open set $U \in \mathbb{K}^n$, a neighbourhood of x and two polynomials P and Q of $\mathbb{K}[X_1, \ldots, x_n]$ such that $Q(R) \neq 0$ and $\forall S \in U$, $f(S) = P(S)/Q(S)$. f is said to be regular on X if it is regular at all points of X.

Thus $\Phi : X \to Y$ is a morphism of algebraic varieties if Φ is continuous on X and if $\forall U \subset Y$ is an open set of Y, and f is a regular map on U, $f \circ \Phi$ is regular on $\Phi^{-1}(U)$. We shall denote by $\Phi^*(f)$, the image of f by the morphism Φ. This equivalence relation yields a coherent definition by 'pasting'.

Let (U, Φ) be a pair formed by an open set U of X and Φ a morphism of U in $Y(U \to Y)$. Consider the following equivalence relation defined by $(U, \Phi) \sim (V, \Psi)$ if $\Phi|_{U\cap V} = \Psi|_{V\cap U}$. We define a *rational mapping* $X \to Y$ as an equivalence class by this relation. The mapping is called *birational* if there also exists a local rational inverse mapping. A *rational function* $X \to \mathbb{K}$ is simply a fraction defined over the whole space. We observe that in projective spaces, rational functions are necessarily constants. Thus it is important to have a local definition for this type of object.

Appendix B. Details of the dP-I parametrization by Weierstrass elliptic functions

B.1. Parametrization

Let t be such that

$$\begin{aligned}(x+y-b) &= \frac{e}{t}\\ (xy-3c) &= t\end{aligned} \tag{B.1}$$

and consider x and y as roots of a second-order equation:

$$X^2 - \left(b+\frac{e}{t}\right)X + (3c+t) = 0. \tag{B.2}$$

Then the solution is

$$(x,y) = \frac{1}{2t}\left[(e+bt) \pm \sqrt{(e+bt)^2 - 4t^2(t+3c)}\right] \tag{B.3}$$

where the discriminant is a polynomial of the order of three in t. This polynomial may be parametrized naturally in terms of Weierstrass elliptic functions. Call δ the quantity

$$\delta = (e+bt)^2 - 4t^2(t+3c). \tag{B.4}$$

We perform a translation $t = z+\alpha$ in δ and choose α in such a way as to have

$$\delta = -4z^3 + g_2 z + g_3. \tag{B.5}$$

This is possible if $\alpha = (\frac{1}{12}b^2 - c)$. Then we have as elliptic Weierstrass invariants:

$$\begin{aligned}g_2 &= 12\left(\tfrac{1}{12}b^2 - c\right)^2 + 2eb\\ g_3 &= -e^2 - 8\left(\tfrac{1}{12}b^2 - c\right)^3 - 2eb\left(\tfrac{1}{12}b^2 - c\right)\end{aligned} \tag{B.6}$$

for the $\mathcal{P}(\xi; g_2, g_3)$ function. If one sets $-z = \mathcal{P}(\xi; g_2, g_3)$, then $\delta = \mathcal{P}'(\xi; g_2, g_3)$. Consequently, the elliptic curve has the parametric representation

$$\begin{aligned}x(\xi) &= \frac{e + b(\frac{1}{12}b^2 - c) - b\mathcal{P}(\xi; g_2, g_3) - \mathcal{P}'(\xi; g_2, g_3)}{2\left(\frac{1}{12}b^2 - e - \mathcal{P}(\xi; g_2, g_3)\right)}\\ y(\xi) &= \frac{e + b(\frac{1}{12}b^2 - c) - b\mathcal{P}(\xi; g_2, g_3) + \mathcal{P}'(\xi; g_2, g_3)}{2\left(\frac{1}{12}b^2 - e - \mathcal{P}(\xi; g_2, g_3)\right)}.\end{aligned} \tag{B.7}$$

Now define v such that $\mathcal{P}(v; g_2, g_3) = (\frac{1}{12}b^2 - c)$. We may then compute the value of the derivative of $\mathcal{P}(\xi; g_2, g_3)$ at $\xi = v$:

$$\mathcal{P}'(\xi; g_2, g_3) = \sqrt{4\mathcal{P}(\xi; g_2, g_3)^3 - g_2\mathcal{P}(\xi; g_2, g_3) - g_3} = e. \tag{B.8}$$

Thus we end up with

$$\begin{aligned}g_2 &= 12\mathcal{P}(v; g_2, g_3)^2 + 2b\mathcal{P}'(v; g_2, g_3)\\ g_3 &= -\mathcal{P}'(v; g_2, g_3)^2 - 8\mathcal{P}(v; g_2, g_3)^3 - 2b\mathcal{P}'(v; g_2, g_3)\,\mathcal{P}(v; g_2, g_3)\}\end{aligned} \tag{B.9}$$

and finally we obtain

$$\begin{aligned}x(\xi) &= \frac{1}{2}\left(b + \frac{\mathcal{P}'(v; g_2, g_3) - \mathcal{P}'(\xi; g_2, g_3)}{\mathcal{P}(v; g_2, g_3) - \mathcal{P}(\xi; g_2, g_3)}\right)\\ y(\xi) &= \frac{1}{2}\left(b + \frac{\mathcal{P}'(v; g_2, g_3) + \mathcal{P}'(\xi; g_2, g_3)}{\mathcal{P}(v; g_2, g_3) - \mathcal{P}(\xi; g_2, g_3)}\right).\end{aligned} \tag{B.10}$$

B.2. Some properties of this parametrization

$x(\xi)$ and $y(\xi)$ may be expressed in terms of the Weierstrass $\varsigma(x)$-function. We make use of the formula

$$\begin{aligned} x(\xi) &= \tfrac{1}{2}b + \varsigma(\xi+v) - \varsigma(\xi) - \varsigma(v) \\ y(\xi) &= \tfrac{1}{2}b + \varsigma(-\xi+v) + \varsigma(\xi) - \varsigma(v). \end{aligned} \tag{B.11}$$

We shall prove (in several steps) that

$$y(\xi+v) = x(\xi). \tag{B.12}$$

For simplicity, from now we denote $\mathcal{P}(u)$ for $\mathcal{P}(u; g_2, g_3)$ and $\mathcal{P}'(u)$ for $\mathcal{P}'(u; g_2, g_3)$.

Step 1. From the parametric representation we compute the sum of $x(\xi)$ and $y(\xi)$:

$$x(\xi) + y(\xi) = b + \frac{\mathcal{P}'(\xi)}{\mathcal{P}(v) - \mathcal{P}(\xi)}. \tag{B.13}$$

Hence

$$x(\xi+v) + y(\xi+v) = b + \frac{\mathcal{P}'(\xi)}{\mathcal{P}(v) - \mathcal{P}(\xi+v)} \tag{B.14}$$

Since $y(\xi+v) = x(\xi)$:

$$x(\xi+v) = -x(\xi) + b + \frac{\mathcal{P}'(\xi)}{\mathcal{P}(v) - \mathcal{P}(\xi+v)}. \tag{B.15}$$

Step 2. Computation of $\mathcal{P}(v) - \mathcal{P}(\xi+v)$ using the addition formula for the $\mathcal{P}(z)$ function:

$$\begin{aligned} \mathcal{P}(\xi+v) &= -\mathcal{P}(\xi) - \mathcal{P}(v) + \frac{1}{4}\left(\frac{\mathcal{P}'(\xi) - \mathcal{P}'(\xi)}{\mathcal{P}(\xi) - \mathcal{P}(v)}\right)^2 \\ &= -\mathcal{P}(\xi) - \mathcal{P}(v) + \left(x(\xi) - \tfrac{1}{2}b\right)^2. \end{aligned} \tag{B.16}$$

Thus we can obtain

$$\mathcal{P}(v) - \mathcal{P}(\xi+v) = \mathcal{P}(\xi) + 2\mathcal{P}(v) - \left(x(\xi) - \tfrac{1}{2}b\right)^2. \tag{B.17}$$

Step 3. We must now compute $\mathcal{P}(\xi) + 2\mathcal{P}(v)$. To this end we form the product

$$\begin{aligned} x(\xi)y(\xi) &= \frac{1}{4}\left[b + \frac{\mathcal{P}'(v) - \mathcal{P}'(\xi)}{\mathcal{P}(v) - \mathcal{P}'(\xi)}\right]\left[b + \frac{\mathcal{P}'(v) + \mathcal{P}'(\xi)}{\mathcal{P}(v) - \mathcal{P}(\xi)}\right] \\ &= \frac{1}{4}b^2 + \frac{1}{2}b\frac{\mathcal{P}'(\xi)}{\mathcal{P}(v) - \mathcal{P}(\xi)} + \frac{1}{4}\frac{\mathcal{P}'(\xi)^2 - \mathcal{P}'(\xi)}{(\mathcal{P}(v) - \mathcal{P}(\xi))^2}. \end{aligned} \tag{B.18}$$

The squares of the derivatives of the $\mathcal{P}(\xi)$ functions may be replaced by polynomials of third order in $\mathcal{P}(\xi)$, and making use of the expression of g_2 in terms of $\mathcal{P}'(\xi)$ and of $\mathcal{P}(v)$ we have

$$\frac{1}{4}\frac{\mathcal{P}'(v)^2 - \mathcal{P}'(\xi)}{(\mathcal{P}(v) - \mathcal{P}(\xi))^2} = -\frac{1}{2}b\frac{\mathcal{P}'(v)}{\mathcal{P}(v) - \mathcal{P}(\xi)} - \mathcal{P}(\xi) - 2\mathcal{P}(v). \tag{B.19}$$

Hence

$$x(\xi)y(\xi) = \tfrac{1}{4}b^2 - \mathcal{P}(\xi) - 2\mathcal{P}(v). \tag{B.20}$$

Consequently

$$\mathcal{P}(v) - \mathcal{P}(\xi+v) = -x(\xi)(x(\xi) + y(\xi) - b) \tag{B.21}$$

and going back to step 1,

$$x(\xi+v) = -x(\xi) + b + \frac{\mathcal{P}'(v)}{-x(\xi)(x(\xi)+y(\xi)-b)}. \tag{B.22}$$

But since the sum $x(\xi) + y(\xi)$ is known from the parametric representation we have

$$x(\xi+v) = -x(\xi) + b + \frac{\mathcal{P}(v) - \mathcal{P}(\xi)}{-x(\xi)}. \tag{B.23}$$

We may now use the expression of $\mathcal{P}(\xi)$ obtained in step 3 in the previous formula and obtained, after simplification, the result

$$x(\xi+v) = -x(\xi) - y(\xi) + b + \frac{3c}{x(\xi)} \tag{B.24}$$

where we have replaced $\frac{1}{4}b^2 - 3\mathcal{P}(v)$ by $3c$. Hence the expected result.

Appendix C. Invariant of the multiplicative perturbation of dP-I: details of the computation

Let us suppose that D_a may be expanded in terms of a: $D_a = \sum_{n\in\mathbb{N}} a^n \Delta_n$ with the obvious property $\phi_a^{h*}(D_a) = D_a$. We will determine D_n order by order so that

$$\phi_a^{h*}\left(\sum_{n=0}^{N} a^n \Delta_n\right) = \sum_{n=0}^{N} a^n \Delta_n + \mathrm{o}(a^N). \tag{C.1}$$

Using the recursion hypothesis we have

$$\phi_a^{h*}\left(\sum_{n=0}^{N-1} a^n \Delta_n\right) = \sum_{n=0}^{N-1} a^n \Delta_n + a^N P_N + \mathrm{o}(a^{N+1}). \tag{C.2}$$

We compute the n^{th} derivative of the *left-hand side* of the previous equation at $\alpha = 0$

$$P_N = \frac{\partial^N}{\partial a^N}\left(\sum_{n=0}^{N-1} a^n \phi_a^{h*}(\Delta_n)\right)_{a=0}. \tag{C.3}$$

Knowing $\Delta_0, \ldots, \Delta_{N-1}$, we obtain D_N as

$$\begin{aligned}\phi_a^{h*}\left(\sum_{n=0}^{N-1} a^n \Delta_n\right) &= \sum_{n=0}^{N-1} a^n \Delta_n + a^N \Delta_N + \mathrm{o}(a^{N+1}) \\ a^N[\phi_a^{h*}(\Delta_N) - \Delta_N] &= -a^N P_N + \mathrm{o}(a^{N+1}).\end{aligned} \tag{C.4}$$

Hence for $a \to 0$ we obtain

$$\phi_0^{h*}(\Delta_N) - \Delta_N = -P_N. \tag{C.5}$$

To obtain Δ_N one must invert the operator $(\phi_0^h - \mathbb{I}d)^*$. This is possible for $|\lambda| < 1$ (outside its kernel). The inverse is then given by the series

$$(\phi_0^h - \mathbb{I}d)^{*(-1)} = \sum_{k\in\mathbb{N}} (\phi_0^{h(k)})^*. \tag{C.6}$$

This yields after simplification

$$P_N = \sum_{k=1}^{N} \frac{1}{k!} \frac{\partial^k}{\partial a^k} \left(\phi_a^{h*}(\Delta_{N-k}) \right)_{a=0}. \tag{C.7}$$

Since the perturbation is linear,

$$\phi_a = \phi_0 + a \left(\frac{\partial \phi_a}{\partial a} \right)_{a=0}$$

and since for the three last components of the transformation ϕ_a we also have

$$\frac{\partial}{\partial a} P_3 \circ \phi_a = \frac{\partial}{\partial a} P_4 \circ \phi_a = \frac{\partial}{\partial a} P_5 \circ \phi_a = 0.$$

Finally, we obtain

$$P_N = \sum_{k=1}^{N} \sum_{p=0}^{k} \frac{(-1)^{k-p}}{p!(k-p)!} t^k v^k \frac{\partial^k}{\partial x^p \partial y^{k-p}} \left(\phi_0^*(\Delta_{N-k}) \right). \tag{C.8}$$

We now show by recursion that

$$\Delta_N(x, y, z, t, v) = v^N t^{N+3} \sum_{a+b+c=2N+3} \frac{\delta_{abc}^N(\lambda)}{x^a y^b z^c} \tag{C.9}$$

where $\delta_{abc}^N(\lambda)$ depends only on λ, let us calculate

$$\frac{\partial^k}{\partial x^p \partial y^{k-p}} \Delta_{N-k} = t^{N-k} v^{N-k+3} \times \sum_{a+b+c=2(N-k)+3} \frac{\delta_{abc}^{N-k}(\lambda)}{x^{c+k} y^{a+p+k-p} z^c} (-1)^k \frac{(a+p-1)!(b+k-p-1)!}{(a-1)!(b-1)!}. \tag{C.10}$$

Letting ϕ_0^* operate on this equation, we have

$$\frac{\partial^k}{\partial x^p \partial y^{k-p}} \phi_0^*(\Delta_{N-k}) = \lambda^{N-k} t^{N-k} v^{N-k+3} (-1)^k p!(k-p)! C_{a+p-1}^{p} C_{b+k-p-1}^{b-p} \times \sum_{a+b+c=2(N-k)+3} \frac{\delta_{abc}^{N-k}(\lambda)}{x^{c+k} y^{a+p} z^{b+k-p}}. \tag{C.11}$$

Now defining new indices:

$$\begin{aligned} a' &= c + k \\ b' &= a + p \\ c' &= b + k - p. \end{aligned} \tag{C.12}$$

Note that $a' + b' + c' = 2N + 3$, we obtain

$$P_N = \sum_{k=1}^{N} \sum_{p=0}^{k} (-1)^p \lambda^N t^N v^{N+3} C_{c'-1}^{b'-p-1} C_{c'-1}^{c'-k+p-1} \frac{\delta_{b'-p, c'-k-p, a'-k}^{N-k}(\lambda)}{x^{a'} y^{b'} c^{c'}}. \tag{C.13}$$

To compute Δ_N we let $(\phi_a - \mathbb{I}d)^{*(-1)}$ act on P_N. But before doing so, we note the sequence

$$\frac{t^N v^{N-k}}{x^{a'} y^{b'} z^{c'}} \underset{\phi_0^*}{\longrightarrow} \lambda^N \frac{t^N v^{N-k}}{x^{c'} y^{a'} z^{b'}} \underset{\phi_0^*}{\longrightarrow} \lambda^{2N} \frac{t^N v^{N-k}}{x^{b'} y^{c'} z^{a'}} \underset{\phi_0^*}{\longrightarrow} \lambda^{3N} \frac{t^N v^{N-k}}{x^{a'} y^{b'} z^{c'}} \underset{\phi_0^*}{\longrightarrow} \cdots. \tag{C.14}$$

We thus obtain D_N as

$$\Delta_N(x,y,z,t,v) = t^N v^{N+3} \sum_{a'+b'+c'=2N+3} \sum_{k=0}^{N} \sum_{p=0}^{k} \frac{(-1)^p \lambda^{N-k}}{1-\lambda^{3N}} \frac{1}{x^{a'} y^{b'} z^{c'}}$$
$$\times \big[C_{b'-1}^{b'-p-1} C_{c'-1}^{c'-k+p-1} \delta^{N-k}_{b'-p,c'-p-k,a'-k}(\lambda)$$
$$+\lambda^N C_{c'-1}^{c'-p-1} C_{a'-1}^{a'-k+p-1} \delta^{N-k}_{c'-p,a'-p-k,b'-k}(\lambda)$$
$$+\lambda^{2N} C_{a'-1}^{a'-p-1} C_{b'-1}^{b'-k+p-1} \delta^{N-k}_{a'-p,b'-p-k,c'-k}(\lambda)\big]. \tag{C.15}$$

$\delta^N_{a',b',c'}(\lambda)$ fulfils the recursion relation:

$$\delta^N_{a',b',c'}(\lambda) = \sum_{k=0}^{N} \sum_{p=0}^{k} \frac{(-1)^p \lambda^{N-k}}{1-\lambda^{3N}} \big[C_{b'-1}^{b'-p-1} C_{c'-1}^{c'-k+p-1} \delta^{N-k}_{b'-p,c'-p-k,a'-k}(\lambda)$$
$$+\lambda^N C_{c'-1}^{c'-p-1} C_{a'-1}^{a'-k+p-1} \delta^{N-k}_{c'-p,a'-p-k,b'-k}(\lambda)$$
$$+\lambda^{2N} C_{a'-1}^{a'-p-1} C_{b'-1}^{b'-k+p-1} \delta^{N-k}_{a'-p,b'-p-k,c'-k}(\lambda)\big]. \tag{C.16}$$

We now show that Δ_N so constructed makes sense, i.e. the series $\sum_{n\in\mathbb{N}} a^n \Delta_n$ is a converging series. Suppose that $\exists c > 0, \exists \mu > 0, \exists N_0, \forall N, \forall a, b, c, \forall N > N_0$:

$$|\delta^N_{a,b,c}(\lambda)| < c\mu^N \tag{C.17}$$

$$|\delta^N_{a,b,c}(\lambda)| \leqslant \frac{|1+\lambda^N+\lambda^{2N}|}{|1-\lambda^{3N}|} \sum_{k=0}^{N} \sum_{p=0}^{k} C_{2(N-2)}^{2(N-2)+p-k} C_{2(N-2)}^{2(N-2)-p} |\lambda\mu|^{N-k} c$$
$$\leqslant c \frac{|1+\lambda^N+\lambda^{2N}|}{|1-\lambda^{3N}|} |\lambda\mu|^N \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} C_{2(N-2)}^{2(N-2)-p} C_{2(N-2)}^{2(N-2)-q} |\lambda\mu|^{-(p+q)}. \tag{C.18}$$

We set $q = k - p$. Since we know that $C_N^k = 0$ for $k < 0$ and using its integral representation, we have

$$|\delta^N_{a,b,c}(\lambda)| \leqslant B_N \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\mathcal{C}(0,\frac{1}{2})} \oint_{\mathcal{C}(0,\frac{1}{2})} (1+w)^{2(n-2)} w^{-2(N-2)-1+2p}$$
$$\times (1+z)^{2(n-2)} z^{-2(N-2)-1+2p} |\lambda\mu|^{-(p+q)} \,\mathrm{d}z\,\mathrm{d}w \tag{C.19}$$

where

$$B_N = c|\lambda\mu|^N \frac{|1+\lambda^N+\lambda^{2N}|}{|1-\lambda^{3N}|}.$$

Note that B_N has a denumerable set of singularities, which is dense on the unit circle in λ. After rearrangement

$$|\delta^N_{a,b,c}(\lambda)| \leqslant B_N \left[\sum_{p=0}^{\infty} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\mathcal{C}(0,\frac{1}{2})} \left(\frac{1+z}{z}\right)^{2(n-2)} \left(\frac{z^2}{|\lambda\mu|}\right)^p \frac{\mathrm{d}z}{z} \right]^2 \tag{C.20}$$

exchanging integration and summation since the series converges, we obtain

$$|\delta^N_{a,b,c}(\lambda)| \leqslant B_N \left[\left(\frac{1+\sqrt{|\lambda\mu|}}{\sqrt{|\lambda\mu|}}\right)^{2(N-2)} - \left(\frac{1-\sqrt{|\lambda\mu|}}{\sqrt{|\lambda\mu|}}\right)^{2(N-2)} \right]^2. \tag{C.21}$$

We can verify that for any N and $|\lambda| > 1$, there exist no solutions μ such that

$$\left(\frac{1+\sqrt{|\lambda\mu|}}{\sqrt{|\lambda\mu|}}\right)^{2N} - \left(\frac{1-\sqrt{|\lambda\mu|}}{\sqrt{|\lambda\mu|}}\right)^{2N} < 2\sqrt{\frac{|1-\lambda^{N+2}|}{|\lambda^{N+2}|}}. \tag{C.22}$$

The *right-hand side* of this expression grows exponentially for $\mu > 0$ even though the other side is bounded by 2 for $|\lambda| > 1$. On the other hand, $\forall|\lambda| < 1$ you can choose μ independent of N when this one is large enough. In fact, as

$$\left(\frac{|1-\sqrt{\mu\lambda}|}{|1+\sqrt{\mu\lambda}|}\right) < 1$$

one may verify that the above expression allows the choice of a lower bound for μ for large N:

$$\mu \geqslant \frac{1}{|\lambda|^{1/2}|1-\lambda^{1/4}|^2}. \tag{C.23}$$

We have thus constructed a converging solution in the complement of a neighbourhood of 0. In fact, for $v = 1$ (i.e. back to a non-homogeneous version), we recall that $\forall\lambda$, $\exists\mu > 0$

$$\begin{aligned} &D_a(x, y, z, t) = \sum_{N\in\mathbb{N}} a^N \Delta_N(x, y, z, t) \\ &\Delta_N(x, y, z, t) = t^N \sum_{a+b+c=2N+3} \frac{\delta^N_{a,b,c}(\lambda)}{x^a y^b z^c} \\ &|\delta^N_{a,b,c}(\lambda)| \leqslant \kappa\mu^N \end{aligned} \tag{C.24}$$

where κ is a positive constant. Thus we have

$$|D_a(x, y, z, t)| \leqslant c \sum_{N\in\mathbb{N}} \mu^N |at|^N \sum_{a+b+c=2N+3} \frac{1}{|x^a y^b z^c|}. \tag{C.25}$$

In the ϕ_λ-invariant manifold $\{x + y + z = e\}$ where e is a constant. We shall prove that

$$\begin{aligned} &D(x, y, t) = \sum_{N\in\mathbb{N}} a^N \Delta_N(x, y, t) \\ &\Delta_N(x, y, t) = t^N \sum_{a+b+c=2N+3} \frac{\delta^N_{a,b,c}(\lambda)}{x^a y^b (e-x-y)^c} \\ &|\delta^N_{a,b,c}(\lambda)| \leqslant \kappa\mu^N. \end{aligned} \tag{C.26}$$

Appendix D. Asymptotic behaviour ($z \to \infty$) of the variables $x(z)$ and $y(z)$

Recall that this change of coordinates is invertible:

$$\begin{aligned} x(\theta) &= \frac{1}{2}\left[\frac{U(\theta)}{\sqrt{z}} + \sqrt{z}\frac{V(\theta)}{U(\theta)}\right] \\ y(\theta) &= \frac{1}{2}\left[\frac{U(\theta)}{\sqrt{z}} - \sqrt{z}\frac{V(\theta)}{U(\theta)}\right]. \end{aligned} \tag{D.1}$$

Hence it follows that

$$xy = \frac{1}{4}\left[\frac{U^2}{z} - z\frac{V^2}{U^2}\right] = \frac{1}{z}\left[\frac{A\sin(\theta-\theta_0)\sin(\theta+\theta_0)}{2}\right]^2 - z\left[\frac{B\cos(\theta+\theta_0)}{A\sin(\theta+\theta_0)}\right]^2. \tag{D.2}$$

Thus

$$\Delta_z(x, y) = (x + y)(xy - z) = \frac{A \sin(\theta - \theta_0) \sin(\theta + \theta_0)}{\sqrt{z}}$$
$$\times \left[\frac{1}{2} \left(\frac{A \sin(\theta - \theta_0) \sin(\theta + \theta_0)}{z} \right)^2 - z \left(1 + \left(\frac{B \cos(\theta + \theta_0)}{A \sin(\theta + \theta_0)} \right)^2 \right) \right]. \quad \text{(D.3)}$$

Recall that the invariants g_2 and g_3 of the Weierstrass function are given by $g_2 = \frac{4}{3}z^2$, $g_3 = \frac{8}{27}z^3 - \Delta^2$. To order $\frac{1}{z}$, as $z \to \infty$, one obtains

$$\frac{3g_3}{2g_2} = \frac{z}{3} - \frac{1}{2}\left(\frac{3\Delta}{2z}\right)^2 = \frac{z}{3} - A^2 \frac{9 \sin^2(\theta - \theta_0)}{8 \sin^2(\theta + \theta_0)} \left[\sin^2(\theta + \theta_0) - \frac{B}{A} \cos^2(\theta + \theta_0) \right]^4 + \mathrm{o}\left(\frac{1}{z}\right). \quad \text{(D.4)}$$

In the autonomous case used we have the parametrization

$$x = \frac{1}{2} \left(\frac{\Delta - \mathcal{P}'(u)}{z/3 + \mathcal{P}(u)} \right)$$
$$y = \frac{1}{2} \left(\frac{\Delta + \mathcal{P}'(u)}{z/3 + \mathcal{P}(u)} \right) \quad \text{(D.5)}$$

or conversely,

$$\mathcal{P}(u) = \frac{2z}{3} - xy = \frac{2z}{3} + z \left(\frac{B \cos(\theta + \theta_0)}{A \sin(\theta + \theta_0)} \right)^2 - \frac{(A \sin(\theta + \theta_0) \sin(\theta - \theta_0))^2}{z^3}. \quad \text{(D.6)}$$

Here the Weierstrass $\mathcal{P}$ function is, to a first approximation, equal to

$$\mathcal{P}(u|g_2, g_3) = -\frac{3g_3}{2g_2} \left(1 - \frac{3}{\phi(u)} \right) \quad \text{(D.7)}$$

with

$$\phi(u) = \sin^2 \sqrt{\frac{3g_3}{2g_2}}$$

since this is one of the cases of degenerate behaviour where one of the periods goes to infinity, i.e. $\Delta = 0$. The identification of these two expressions of $\mathcal{P}(u)$ yields

$$\frac{1}{\phi} = \left(\frac{2}{3}z - xy + \frac{3g_3}{2g_2} \right) \frac{2g_2}{3g_3} \quad \text{(D.8)}$$

but explicit computations of the *right-hand side* of this expression give

$$\frac{1}{\phi} = z \left[1 + \left(\frac{B \cos(\theta + \theta_0)}{A \sin(\theta + \theta_0)} \right)^2 \right] - \frac{A^2}{4z} \sin^2(\theta + \theta_0) \sin^2(\theta - \theta_0)$$
$$- \frac{9A^2}{8z} \frac{\sin^2(\theta - \theta_0)}{\sin^2(\theta + \theta_0)} \left(\sin^2(\theta + \theta_0) + \frac{B^2}{A^2} \cos^2(\theta + \theta_0) \right) + \mathrm{o}\left(\frac{1}{z}\right) \quad \text{(D.9)}$$

we obtain

$$\Phi = \frac{\sin^2(\theta + \theta_0)}{1 + \left(B^2/A^2 - 1\right) \cos^2(\theta + \theta_0)}. \quad \text{(D.10)}$$

As in the theory of differential equations, we proceed with the method of variation of the constant. We may suppose that A/B depends on z such that

$$\Phi = \frac{\sin^2(\theta+\theta_0)}{1+\left[(A(z)/B(z))^2-1\right]\cos^2(\theta+\theta_0)} + \mathrm{o}\left(\frac{1}{z}\right). \tag{D.11}$$

The plot of the quantity $\frac{A(z)}{B(z)}$ as a function of z reveals that

$$\lim_{z\to\infty}\left(\frac{A(z)}{B(z)}\right)^3 = \lim_{z\to\infty} 4\left(\frac{U(z)}{V(z)}\right)^2\left(\frac{1}{\Phi(z)}-1\right) = 1. \tag{D.12}$$

This is consistent with the degeneracy of $\mathcal{P}(u|g_2, g_3)$. Another degeneracy occurs when two other roots of $\mathcal{P}(u|g_2, g_3)$ are equal. This situation may be obtained by application of F_ϵ^*.

We shall show that $\lim_{z\to\infty}\frac{A(z)}{B(z)} = 1$, and the temporal dependence is in $\ln z$.

To this end, note that $F_\epsilon^{(2)}$ is a rotation of angle $\frac{\pi}{2}+\alpha(z)$ such that $\alpha(z) \ll \frac{\pi}{2} \bmod 2\pi$. From the parity of Φ we deduce that $F_\epsilon^{(4)}$ is a rotation of angle $\alpha(z)+\alpha(z+2\epsilon)$. Hence using the expression of U and V we have

$$\begin{aligned} F^{(2)*}(U(\theta)) &= A(z+2\epsilon)\cos(\theta+\theta_0+\alpha(z))\cos(\theta-\theta_0+\alpha(z)) \\ F^{(2)*}(V(\theta)) &= -2B(z+2\epsilon)\cos(\theta+\theta_0+\alpha(z))\sin(\theta-\theta_0+\alpha(z)). \end{aligned} \tag{D.13}$$

But as one has the sequence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow[F^{(2)}]{} \begin{bmatrix} y - \dfrac{2}{x} - \dfrac{x(z+\epsilon)}{x(x+y)-z} \\ -x-y+\dfrac{z}{x} \\ z+2\epsilon \end{bmatrix} \tag{D.14}$$

we obtain

$$\begin{aligned} F^{(2)*}(y) &= -x-y+\frac{z}{x} \\ F^{(2)*}(x+y)) &= \frac{-x(x(x+y)+\epsilon)}{x(x+y)-(z+\epsilon)}. \end{aligned} \tag{D.15}$$

Plugging back into the initial expressions we obtain two equations:

Equation I:

$$\begin{aligned} &\frac{B\sqrt{z+2\epsilon}}{A}\tan(\theta+\theta_0+\alpha) + \frac{A}{2\sqrt{z+2\epsilon}}\cos(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha) \\ &\quad = -\frac{A}{\sqrt{z}}\sin(\theta+\theta_0) \\ &\quad\quad \times\left[\sin(\theta-\theta_0) - \frac{2z^2}{A^2\sin^2(\theta+\theta_0)\sin(\theta-\theta_0)+2Bz\cos(\theta+\theta_0)}\right]. \end{aligned} \tag{D.16}$$

Equation II:

$$\frac{A}{\sqrt{(z+2\epsilon)}}\cos(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha)$$

$$= -\frac{A^2 \sin^2(\theta+\theta_0)\sin(\theta-\theta_0) + 2Bz\cos(\theta+\theta_0)}{2A\sqrt{z}\sin(\theta+\theta_0)}$$
$$\times \left[1 + \frac{2z(z+\epsilon)}{A^2\sin^2(\theta+\theta_0)\sin(\theta-\theta_0) + 2Bz\sin(\theta+\theta_0)\cos(\theta+\theta_0) - 2z^2}\right]. \qquad \text{(D.17)}$$

Performing the product of the two equations, we find

$$\frac{A}{\sqrt{(z+2\epsilon)}}\cos(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha)$$
$$\times\left[\frac{A}{\sqrt{(z+2\epsilon)}}\cos(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha) + B\sqrt{\frac{z+2\epsilon}{A}}\frac{\sin(\theta+\theta_0+\alpha)}{\cos(\theta+\theta_0+\alpha)}\right]$$
$$= \frac{1}{2z}\left[A^2\sin^2(\theta+\theta_0)\sin^2(\theta-\theta_0) + 2Bz\sin(\theta-\theta_0)\cos(\theta+\theta_0) + 2\epsilon z\right]. \qquad \text{(D.18)}$$

Assuming that B is a constant to order $\mathrm{o}\left(\frac{1}{z}\right)$ we set

$$u = \frac{z}{B}$$
$$g = \frac{\epsilon}{B} \qquad \text{(D.19)}$$
$$r = \left(\frac{B}{A}\right)^2$$

and feed back to this order into equations I and II. They become to order $\mathrm{o}\left(\frac{1}{z}\right)$.

Equation I:

$$r(u+2g)\tan(\theta+\theta_0+\alpha) + \tfrac{1}{2}\cos(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha) = -\sqrt{\frac{u+2g}{u}}\sin(\theta+\theta_0)$$
$$\times\left[\sin(\theta-\theta_0) - \frac{2ru^2}{\sin^2(\theta+\theta_0)\sin^2(\theta-\theta_0) + 2ru\cos(\theta+\theta_0)}\right]. \qquad \text{(D.20)}$$

Equation II:

$$\cos(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha) - 2r(u+2g)\sin(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha)$$
$$= \sqrt{\frac{u+2g}{u}}\left[\sin^2(\theta+\theta_0)\sin^2(\theta-\theta_0) + 2ru\sin(\theta-\theta_0)\cos(\theta+\theta_0) + 2gru\right]. \qquad \text{(D.21)}$$

Identification will yield relations between g, θ_0, α and r for any θ. Expanding and regrouping according to powers of u we have the following.

Equation I:

$$u^2[4r^2\cos(\theta+\theta_0)\sin(\theta+\theta_0+\alpha) - 4r\cos(\theta+\theta_0+\alpha)\sin(\theta+\theta_0)]$$
$$+u[2r\cos(\theta+\theta_0)\cos^2(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha)$$
$$+2r\sin^2(\theta+\theta_0)\sin(\theta-\theta_0)\sin(\theta+\theta_0+\alpha)$$

$$+8gr^2 \cos(\theta+\theta_0)\sin(\theta+\theta_0+\alpha)$$
$$+4r\cos(\theta+\theta_0)\cos(\theta+\theta_0+\alpha)\sin(\theta+\theta_0)\sin(\theta-\theta_0)$$
$$-4gr\cos(\theta+\theta_0+\alpha)\sin(\theta+\theta_0)]$$
$$+u^0[\sin^2(\theta+\theta_0)\sin(\theta-\theta_0)\cos(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha)$$
$$+4gr\sin^2(\theta+\theta_0)\sin(\theta-\theta_0)\sin(\theta+\theta_0+\alpha)$$
$$+2\sin^2(\theta+\theta_0)\sin^2(\theta-\theta_0)\cos(\theta+\theta_0+\alpha)$$
$$-4gr\cos(\theta+\theta_0)\sin(\theta+\theta_0)\sin(\theta-\theta_0)\cos(\theta+\theta_0+\alpha)$$
$$+4g^2r\cos(\theta+\theta_0+\alpha)\sin(\theta+\theta_0)] + \mathrm{o}\left(\frac{1}{u}\right) = 0. \tag{D.22}$$

Equation II:

$$u^2[g+\sin(\theta-\theta_0)\cos(\theta+\theta_0) - \sin(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha)]$$
$$+u[\sin^2(\theta+\theta_0)\sin^2(\theta-\theta_0) - \cos^2(\theta+\theta_0+\alpha)\cos^2(\theta-\theta_0+\alpha)$$
$$+4gr(g+\sin(\theta-\theta_0)\cos(\theta+\theta_0) - \sin(\theta+\theta_0+\alpha)\cos(\theta+\theta_0+\alpha)]$$
$$+u^0[2g\sin^2(\theta-\theta_0)\sin^2(\theta-\theta_0)] + \mathrm{o}\left(\frac{1}{u}\right) = 0. \tag{D.23}$$

Cancellation to order u^2 gives the conditions

$$\begin{aligned} &r\cos(\theta+\theta_0)\sin(\theta+\theta_0+\alpha) = \cos(\theta+\theta_0+\alpha)\sin(\theta+\theta_0) \\ &g+\sin(\theta-\theta_0)\cos(\theta+\theta_0) = \sin(\theta+\theta_0+\alpha)\cos(\theta-\theta_0+\alpha). \end{aligned} \tag{D.24}$$

If r, α, g are to be independent of θ one must choose

$$\begin{aligned} &r = 1 \\ &\alpha = 0 \\ &g = 2\sin(\theta_0). \end{aligned} \tag{D.25}$$

We proceed to the next order by setting

$$\begin{aligned} &r = 1 + \frac{a}{u} \\ &\alpha = \frac{b}{u} \\ &g = 2\sin(\theta_0) + \frac{c}{u} \end{aligned} \tag{D.26}$$

where a, b, c are parameters to be determined. Feeding this new input into the equations and expanding them from order u^2 to u^{-1}, we obtain after expansion of trigonometric functions and simplifications:

Equation I:

$$2[2b+\cos(2\theta_0)] + \sin 2(\theta+\theta_0)[2a+\cos(2\theta_0)] = 0. \tag{D.27}$$

Equation II:

$$c + [2b + \cos(2\theta_0)] = 0. \tag{D.28}$$

The simultaneous solutions to the two necessary conditions are obviously

$$\begin{aligned} a &= -\frac{\sin(2\theta_0)}{2} \\ b &= -\frac{\cos(2\theta_0)}{2} \\ c &= 0. \end{aligned} \tag{D.29}$$

To sum up, we have obtained

$$\begin{aligned} r &= 1 - \frac{\sin(2\theta_0)}{2u} + \mathrm{o}\left(\frac{1}{u^2}\right) \\ \alpha &= -\frac{\cos(2\theta_0)}{2u} + \mathrm{o}\left(\frac{1}{u^2}\right) \\ g &= \sin(2\theta_0) + \mathrm{o}\left(\frac{1}{u^2}\right). \end{aligned} \tag{D.30}$$

This solution is surprising since the system of equations is overdetermined with respect to the number of variables. Such a situation usually arises in integrable systems and one might state that the system under study may be integrable. With these expressions, one may show that $r \to 1$ as $z \to \infty$ and U, V depend on $\ln z$. The previous expansion may also be extended to higher orders in $\left(\frac{1}{u}\right)$. For the sake of consistency we shall use an expansion of the Weierstrass-$\mathcal{P}$ function. Recall that [4]

$$\mathcal{P}(u|g_2, g_3) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(u\sqrt{e_1 - e_3}, k)} \tag{D.31}$$

where e_1, e_2 and e_3 are roots of $4x^3 - g_2 x - g_3 = 0$, and $\mathrm{sn}(u, k)$ is the Jacobian elliptic sine function of modulus

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}.$$

Moreover, one has the following expansion of the inverse of $\sin^2(u, k)$:

$$\frac{1}{\mathrm{sn}^2(u, k)} = \left(\frac{\pi}{2K}\right)^2 \frac{1}{\sin^2(u\pi/2K)} + \frac{K - E}{K} - \frac{2\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos\left(\frac{n\pi}{K} u\right). \tag{D.32}$$

Here K and E are complete elliptic integrals[4].

Now since

$$\alpha = -\frac{\cos(2\theta_0)}{2u} + \mathrm{o}\left(\frac{1}{u^2}\right)$$

[4] $K = K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} k^{2n} \left(\frac{(2n-1)!!}{2^n n}\right)^2$

$E = E(k) = \frac{\pi}{2} \left[1 - \sum_{n=0}^{\infty} k^{2n} \left(\frac{(2n-1)!!}{2^n n}\right)^2\right]$

$q = \mathrm{e}^{-\pi \frac{K(k)}{K(k')}}$ with $k^2 + k'^2 = 1$.

then

$$\sum_{z\longmapsto z+2\epsilon}\alpha(u(z))=\sum_{u\longmapsto u+2g}^{u}-\frac{\cos(2\theta_0)}{2u}\simeq\phi-\frac{\cos(2\theta_0)}{4g}\ln\frac{u}{2g}+\mathrm{o}\left(\frac{1}{u}\right)$$

$$\simeq\phi-\frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon}+\mathrm{o}\left(\frac{1}{z}\right)\tag{D.33}$$

where ϕ is a constant.

Thus in the limit $z\to\infty$, one obtains the asymptotic expression of U and V, and the equivalent ones for x and y, assuming that we can express θ_0 as a function of ϕ and depending on the initial conditions:

$$\begin{aligned}U(z)&=A\sin\left[\phi-\frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon}+\theta_0\right]\sin\left[\phi-\frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon}-\theta_0\right]\\V(z)&=2A\cos\left[\phi-\frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon}+\theta_0\right]\sin\left[\phi-\frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon}-\theta_0\right]\\x(z)&=2\sqrt{z}\cot\left[\phi-\frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon}+\theta_0\right]\\y(z)&=-2\sqrt{z}\cot\left[\phi-\frac{1}{8\sin(\theta_0)}\ln\frac{z}{2\epsilon}+\theta_0\right]\end{aligned}\tag{D.34}$$

where A and ϕ depend on initial conditions as it is usual in the resolution of differential equations: $A_{x,y}$ and $\phi_{x,y}$.

References

[1] Ablowitz M J, Halburd R and Herbst B 2000 On the extension of Painlevé property to difference equations *Nonlinearity* **13** 889–905

[2] Arnol'd V I 1990 Dynamics of complexity of intersections *Bol. Soc. Bras. Mat.* **21** 1–10

[3] Bellon M P and Viallet C-M 1999 Algebraic entropy *Commun. Math. Phys.* **204** 425–37

[4] Bessis D, Itzykson C and Zuber J B 1980 Quantum field theory in graphical enumeration *Adv. Appl. Math.* **1** 109–57

[5] Boukraa S, Maillard J-M and Rollet G 1994 Integrable mappings and polynomial growth *Physica* A **209** 162–222

[6] Broer H W, Huitema G B and Sevryuk M B 1996 *Quasi-Periodic Motions in Families of Dynamical Systems (Lectures Notes in Mathematics vol 1645)* (Berlin: Springer)

[7] Conte R and Musette M 1998 *Rules of Discretization for Painlevé Equations* in *Theory of Nonlinear Special Functions: The Painlevé Transcendents* ed L Vinet and Winternitz P (Berlin: Springer)
(Conte R and Musette M 1998 *Preprint* solv-int/980305)

[8] Erdelyi A (ed) 1989 *Higher Transcendental Functions* vol II (New York: Krieger)

[9] Falqui G and Viallet C-M 1993 Singularity, complexity and quasi-integrability of rational mappings *Commun. Math. Phys.* **154** 111–25

[10] Fokas A S, Its A R and Kitaev A V 1991 Discrete Painlevé equations and their appearance in quantum gravity *Commun. Math. Phys.* **142** 313–44

[11] Grammaticos B, Nijhoff F W and Ramani A 1999 *Discrete Painlevé equations. The Painlevé property (CRM Series in Mathematics Physics)* (New York: Springer) pp 413–516

[12] Grammaticos B, Ramani A and Papageorgiou V 1991 Do integrable mappings have the Painlevé property *Phys. Rev. Lett.* **67** 1825

[13] Grammaticos B, Hietarinta J and Ramani A 1991 Discrete versions of the Painlevé equations *Phys. Rev. Lett.* **67** 1829–32

[14] Hietarinta J and Viallet C M 2000 Singularity confinement and chaos in discrete systems *Phys. Rev. Lett.* **81** 325

[15] Hietarinta J and Viallet C M 1998 *Singularity Confinement and Degree Growth (CMR Proc. and Lecture Notes vol 25)* (Providence, RI: American Mathematical Society)

[16] Hietarinta J and Viallet C M 2000 Discrete Painlevé and singularity confinement in projective space *Chaos Solitons Fractals* **11** 29–32

[17] Innes R T A 1906–7 On the periods of the Elliptic functions of Weierstrass *Proc. R. Soc. Edin.* **27** 357–68

[18] Joshi N 1997 A local asymptotic analysis of the discrete first Painlevé equation as the discrete independent variable approaches infinity (dedicated to Martin David Kruskal) *Methods Appl. Anal.* **4** 124–33

[19] Joshi N 2000 Irregular singular behaviour in the first discrete Painlevé equation *SIDE III: Symmetries and Integrability of Difference Equations (Sabaudia, 1998) (CRM Proc. Lecture Notes 25)* (Providence, RI: American Mathematical Society) pp 237–43

[20] Ohta Y, Tamizhmani K M, Grammaticos B and Ramani A 1999 Singularity confinement and algebraic entropy: the case of the discrete Painlevé equations *Phys. Lett.* A **262** 152–7

[21] Okamoto K 1999 *The Hamiltonians Associated to the Painlevé Equations. The Painlevé Property (CRM Series in Mathematics Physics)* (Berlin: Springer) pp 735–87

[22] Quispel G R, Roberts J A G and Thompson C J 1989 Integrable mappings and soliton equations. II *Physica* D **34** 183–92

[23] Shafarevich I R 1994 *Basic Algebraic Geometry. 1, Varieties in Projective Space* (Berlin: Springer)

[24] Shohat J 1934 A differential equation for orthogonal polynomials *Duke J. Math.* **5** 401–17